Introduction to Vector Spaces, Vector Algebras, and Vector Geometries

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Abstract

An introductory overview of vector spaces, algebras, and linear geometries over an arbitrary commutative field is given. Quotient spaces are emphasized and used in constructing the exterior and the symmetric algebras of a vector space. The exterior algebra of a vector space and that of its dual are used in treating linear geometry. Scalar product spaces, orthogonality, and the Hodge star based on a general basis are treated.

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1 Fundamentals of Structure

1.1 What is a Vector Space?

A vector space is a structured set of elements called vectors. The structure required for this set of vectors is that of an Abelian group with operators from a specified field. (For us, multiplication in a field is commutative, and a number of our results will depend on this in an essential way.) If the specified field is \mathcal{F} , we say that the vector space is **over** \mathcal{F} . It is standard for the Abelian group to be written additively, and for its identity element to be denoted by 0. The symbol 0 is also used to denote the field's zero element, but using the same symbol for both things will not be confusing. The field elements will operate on the vectors by multiplication on the left, the result always being a vector. The field elements will be called **scalars** and be said to **scale** the vectors that they multiply. The scalars are required to scale the vectors in a harmonious fashion, by which is meant that the following three rules always hold for any vectors and scalars. (Here a and b denote scalars, and b and b denote vectors.)

- 1. Multiplication by the field's unity element is the identity operation: $1 \cdot v = v$.
- 2. Multiplication is associative: $a \cdot (b \cdot v) = (ab) \cdot v$.
- 3. Multiplication distributes over addition: $(a + b) \cdot v = a \cdot v + b \cdot v$ and $a \cdot (v + w) = a \cdot v + a \cdot w$.

Four things which one might reasonably expect to be true are true.

Proposition 1 If a is a scalar and v a vector, $a \cdot 0 = 0$, $0 \cdot v = 0$, $(-a) \cdot v = -(a \cdot v)$, and given that $a \cdot v = 0$, if $a \neq 0$ then v = 0.

Proof: $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$; adding $-(a \cdot 0)$ to both ends yields the first result. $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$; adding $-(0 \cdot v)$ to both ends yields the second result. Also, $0 = 0 \cdot v = (a + (-a)) \cdot v = a \cdot v + (-a) \cdot v$; adding $-(a \cdot v)$ on the front of both ends yields the third result. If $a \cdot v = 0$ with $a \neq 0$, then $a^{-1} \cdot (a \cdot v) = a^{-1} \cdot 0$, so that $(a^{-1}a) \cdot v = 0$ and hence v = 0 as claimed. \blacksquare

Exercise 1.1 Given that $a \cdot v = 0$, if $v \neq 0$ then a = 0.

Exercise 1.2 $-(a \cdot v) = a \cdot (-v)$.

Because of the harmonious fashion in which the scalars operate on the vectors, the elementary arithmetic of vectors contains no surprises.

1.2 Some Vector Space Examples

There are many familiar spaces that are vector spaces, although in their usual appearances they may have additional structure as well. A familiar vector space over the real numbers \mathbb{R} is \mathbb{R}^n , the space of n-tuples of real numbers with component-wise addition and scalar multiplication by elements of \mathbb{R} . Closely related is another vector space over \mathbb{R} , the space $\Delta \mathbb{R}^n$ of all translations of \mathbb{R}^n , the general element of which is the function from \mathbb{R}^n to itself that sends the general element $(\xi_1,...,\xi_n)$ of \mathbb{R}^n to $(\xi_1+\delta_1,...,\xi_n+\delta_n)$ for a fixed real *n*-tuple $(\delta_1, ..., \delta_n)$. Thus an element of $\Delta \mathbb{R}^n$ uniformly vectors the elements of \mathbb{R}^n to new positions, displacing each element of \mathbb{R}^n by the same vector, and in this sense $\Delta \mathbb{R}^n$ is truly a space of vectors. In the same fashion as with \mathbb{R}^n over \mathbb{R} , over any field \mathcal{F} , the space \mathcal{F}^n of all n-tuples of elements of \mathcal{F} constitutes a vector space. The space \mathbb{C}^n of all n-tuples of complex numbers in addition to being considered to be a vector space over \mathbb{C} may also be considered to be a (different, because it is taken over a different field) vector space over \mathbb{R} . Another familiar algebraic structure that is a vector space, but which also has additional structure, is the space of all polynomials with coefficients in the field \mathcal{F} . To conclude this initial collection of examples, the set consisting of the single vector 0 may be considered to be a vector space over any field; these are the **trivial** vector spaces.

1.3 Subspace, Linear Combination and Span

Given a vector space \mathcal{V} over the field \mathcal{F} , \mathcal{U} is a **subspace** of \mathcal{V} , denoted $\mathcal{U} \lhd \mathcal{V}$, if it is a subgroup of \mathcal{V} that is itself a vector space over \mathcal{F} . To show that a subset \mathcal{U} of a vector space is a subspace, it suffices to show that \mathcal{U} is closed under sum and under product by any scalar. We call \mathcal{V} and $\{0\}$ improper subspaces of the vector space \mathcal{V} , and we call all other subspaces **proper**.

Exercise 1.3 As a vector space over itself, \mathbb{R} has no proper subspace. The set of all integers is a subgroup, but not a subspace.

Exercise 1.4 The intersection of any number of subspaces is a subspace.

A linear combination of a finite set S of vectors is any sum $\sum_{s \in S} c_s \cdot s$ obtained by adding together exactly one multiple, by some scalar coefficient, of each vector of S. A linear combination of an infinite set S of vectors is a linear combination of any finite subset of S. The empty sum which results in forming the linear combination of the empty set is taken to be 0, by convention. A subspace must contain the value of each of its linear combinations. If S is any subset of a vector space, by the **span** of S, denoted S, is meant the set of the values of all linear combinations of S. S is a subspace. If S is a spanning set for S.

Exercise 1.5 For any set S of vectors, $\langle S \rangle$ is the intersection of all subspaces that contain S, and therefore $\langle S \rangle$ itself is the only subspace of $\langle S \rangle$ that contains S.

1.4 Independent Set, Basis and Dimension

Within a given vector space, a **dependency** is said to exist in a given set of vectors if the zero vector is the value of a nontrivial (scalar coefficients not all zero) linear combination of some finite nonempty subset of the given set. A set in which a dependency exists is (**linearly**) **dependent** and otherwise is (**linearly**) **independent**. The "linearly" can safely be omitted and we will do this to simplify our writing.

For example, if $v \neq 0$, then $\{v\}$ is an independent set. The empty set is an independent set. Any subset of an independent set is independent. On the other hand, $\{0\}$ is dependent, as is any set that contains 0.

Exercise 1.6 A set S of vectors is independent if and only if no member is in the span of the other members: for all $v \in S$, $v \notin \langle S \setminus \{v\} \rangle$. What would the similar statement need to be if the space were not a vector space over a field, but only the similar type of space over the ring \mathbb{Z} of ordinary integers?

Similar ideas of dependency and independence may be introduced for vector sequences. A vector sequence is **dependent** if the set of its terms is a dependent set or if the sequence has any repeated terms. Otherwise the sequence is **independent**.

Exercise 1.7 The terms of the finite vector sequence v_1, \ldots, v_n satisfy a linear relation $a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0$ with at least one of the scalar coefficients a_i nonzero if and only if the sequence is dependent. If the sequence is dependent, then it has a term that is in the span of the set of preceding terms and this set of preceding terms is nonempty if $v_1 \neq 0$.

A maximal independent set in a vector space, i. e., an independent set that is not contained in any other independent set, is said to be a **basis** set, or, for short, a **basis** (plural: **bases**), for the vector space. For example, $\{(1)\}$ is a basis for the vector space \mathcal{F}^1 , where \mathcal{F} is any field. The empty set is the unique basis for a trivial vector space.

Exercise 1.8 Let Φ be a family of independent sets which is linearly ordered by set inclusion. Then the union of all the sets in Φ is an independent set.

From the result of the exercise above and Zorn's Lemma applied to independent sets which contain a given independent set, we obtain the following important result.

Theorem 2 Every vector space has a basis, and, more generally, every independent set is contained in some basis. ■

A basis may be characterized in various other ways besides its definition as a maximal independent set.

Proposition 3 (Independent Spanning Set) A basis for the vector space V is precisely an independent set that spans V.

Proof: In the vector space \mathcal{V} , let \mathcal{B} be an independent set such that $\langle \mathcal{B} \rangle = \mathcal{V}$. Suppose that \mathcal{B} is contained in the larger independent set \mathcal{C} . Then, choosing any $v \in \mathcal{C} \setminus \mathcal{B}$, the set $\mathcal{B}' = \mathcal{B} \cup \{v\}$ is independent because it is a subset of the independent set \mathcal{C} . But then $v \notin \langle \mathcal{B}' \setminus \{v\} \rangle$, i. e., $v \notin \langle \mathcal{B} \rangle$, contradicting $\langle \mathcal{B} \rangle = \mathcal{V}$. Hence \mathcal{B} is a basis for \mathcal{V} .

On the other hand, let \mathcal{B} be a basis, i. e., a maximal independent set, but suppose that \mathcal{B} does not span \mathcal{V} . There must then exist $v \in \mathcal{V} \setminus \langle \mathcal{B} \rangle$. However, $\mathcal{B} \cup \{v\}$ would then be independent, since a nontrivial dependency relation in $\mathcal{B} \cup \{v\}$ would have to involve v with a nonzero coefficient and this would put $v \in \langle \mathcal{B} \rangle$. But the independence of $\mathcal{B} \cup \{v\}$ contradicts the hypothesis that \mathcal{B} is a maximal independent set. Hence, the basis \mathcal{B} is an independent set that spans \mathcal{V} .

Corollary 4 An independent set is a basis for its own span. ■

Proposition 5 (Minimal Spanning Set) \mathcal{B} is a basis for the vector space \mathcal{V} if and only if \mathcal{B} is a minimal spanning set for \mathcal{V} , i. e., \mathcal{B} spans \mathcal{V} , but no subset of \mathcal{B} unequal to \mathcal{B} also spans \mathcal{V} .

Proof: Suppose \mathcal{B} spans \mathcal{V} and no subset of \mathcal{B} unequal to \mathcal{B} also spans \mathcal{V} . If \mathcal{B} were not independent, there would exist $b \in \mathcal{B}$ for which $b \in \langle \mathcal{B} \setminus \{b\} \rangle$. But then the span of $\mathcal{B} \setminus \{b\}$ would be the same as the span of \mathcal{B} , contrary to what we have supposed. Hence \mathcal{B} is independent, and by the previous proposition, \mathcal{B} must therefore be a basis for \mathcal{V} since \mathcal{B} also spans \mathcal{V} .

On the other hand, suppose \mathcal{B} is a basis for the vector space \mathcal{V} . By the previous proposition, \mathcal{B} spans \mathcal{V} . Suppose that \mathcal{A} is a subset of \mathcal{B} that is unequal to \mathcal{B} . Then no element $v \in \mathcal{B} \setminus \mathcal{A}$ can be in $\langle \mathcal{B} \setminus \{v\} \rangle$ since \mathcal{B} is independent, and certainly, then, $v \notin \langle \mathcal{A} \rangle$ since $\mathcal{A} \subset \mathcal{B} \setminus \{v\}$. Thus \mathcal{A} does not span \mathcal{V} , and \mathcal{B} is therefore a minimal spanning set for \mathcal{V} .

Exercise 1.9 Consider a finite spanning set. Among its subsets that also span, there is at least one of smallest size. Thus a basis must exist for any vector space that has a finite spanning set, independently verifying what we already know from Theorem 2.

Exercise 1.10 Over any field \mathcal{F} , the set of n n-tuples that have a 1 in one position and 0 in all other positions is a basis for \mathcal{F}^n . (This basis is called the **standard basis** for \mathcal{F}^n over \mathcal{F} .)

Proposition 6 (Unique Linear Representation) \mathcal{B} is a basis for the vector space \mathcal{V} if and only if \mathcal{B} has the unique linear representation property, i. e., each vector of \mathcal{V} has a unique linear representation as a linear combination $\sum_{x \in \mathcal{X}} a_x \cdot x$ where \mathcal{X} is some finite subset of \mathcal{B} and all of the scalars a_x are nonzero.

Proof: Let \mathcal{B} be a basis for \mathcal{V} , and let $v \in \mathcal{V}$. Since \mathcal{V} is the span of \mathcal{B} , v certainly is a linear combination of a finite subset of \mathcal{B} with all scalar coefficients nonzero. Suppose that v has two different expressions of this kind. Subtracting, we find that 0 is equal to a nontrivial linear combination of the union \mathcal{U} of the subsets of \mathcal{B} involved in the two expressions. But \mathcal{U} is an independent set since it is a subset of the independent set \mathcal{B} . Hence v has the unique representation as claimed.

On the other hand, suppose that \mathcal{B} has the unique linear representation property. Then \mathcal{B} spans \mathcal{V} . We complete the proof by showing that \mathcal{B} must be an independent set. Suppose not. Then there would exist $b \in \mathcal{B}$ for which $b \in \langle \mathcal{B} \setminus \{b\} \rangle$. But this b would then have two different representations with nonzero coefficients, contrary to hypothesis, since we always have $1 \cdot b$ as one such representation and there would also be another one not involving b in virtue of $b \in \langle \mathcal{B} \setminus \{b\} \rangle$.

Exercise 1.11 A finite set of vectors is a basis for a vector space if and only if each vector in the vector space has a unique representation as a linear combination of this set: $\{x_1, \ldots, x_n\}$ (with distinct x_i , of course) is a basis if and only if each $v = a_1 \cdot x_1 + \cdots + a_n \cdot x_n$ for unique scalars a_1, \ldots, a_n .

Exercise 1.12 If S is a finite independent set, $\sum_{s \in S} c_s \cdot s = \sum_{s \in S} d_s \cdot s$ implies $c_s = d_s$ for all s.

Example 7 In this brief digression we now apply the preceding two propositions. Let v_0, v_1, \ldots, v_n be vectors in a vector space over the field \mathcal{F} , and suppose that v_0 is in the span \mathcal{V} of the other v_i . Then the equation

$$\xi_1 \cdot v_1 + \cdots + \xi_n \cdot v_n = v_0$$

for the $\xi_i \in \mathcal{F}$ has at least one solution. Renumbering the v_i if necessary, we may assume that the distinct vectors v_1, \ldots, v_m form a basis set for \mathcal{V} . If m = n the equation has a unique solution. Suppose, however, that $1 \leq m < n$. Then the equation

$$\xi_1 \cdot v_1 + \dots + \xi_m \cdot v_m = v_0 - \xi_{m+1} \cdot v_{m+1} - \dots - \xi_n \cdot v_n$$

has a unique solution for ξ_1, \ldots, ξ_m for each fixed set of values we give the other ξ_i , since the right-hand side is always in \mathcal{V} . Thus ξ_1, \ldots, ξ_m are functions of the variables ξ_{m+1}, \ldots, ξ_n , where each of these variables is allowed to range freely over the entire field \mathcal{F} . When \mathcal{F} is the field \mathbb{R} of real numbers, we have deduced a special case of the Implicit Function Theorem of multivariable calculus. When the v_i are d-tuples (i. e., elements of \mathcal{F}^d), the original vector equation is the same thing as the general set of d consistent numerical linear equations in n unknowns, and our result describes how, in the general solution, n-m of the ξ_i are arbitrary parameters upon which the remaining m of the ξ_i depend.

We have now reached a point where we are able to give the following key result.

Theorem 8 (Replacement Theorem) Let V be a vector space with basis \mathcal{B} , and let \mathcal{C} be an independent set in V. Given any finite subset of \mathcal{C} no larger than \mathcal{B} , its elements can replace an equal number of elements of \mathcal{B} and the resulting set will still be a basis for V.

Proof: The theorem holds in the case when no elements are replaced. Suppose that the elements $y_1, ..., y_N$ of \mathcal{C} have replaced N elements of \mathcal{B} and the resulting set \mathcal{B}_N is still a basis for \mathcal{V} . An element y_{N+1} of $\mathcal{C} \setminus \{y_1, ..., y_N\}$ has a unique linear representation as a linear combination with nonzero coefficients of some finite nonempty subset $\mathcal{X} = \{x_1, ..., x_K\}$ of \mathcal{B}_N . There must be an element $x^* \in \mathcal{X} \setminus \{y_1, ..., y_N\}$ because y_{N+1} cannot be a linear combination of $\{y_1, ..., y_N\}$ since \mathcal{C} is independent. In \mathcal{B}_N replace x^* with y_{N+1} . Clearly, the result \mathcal{B}_{N+1} still spans \mathcal{V} .

The assumption that $x \in \mathcal{B}_{N+1}$ is a linear combination of the other elements of \mathcal{B}_{N+1} will be seen to always contradict the independence of \mathcal{B}_N , proving that \mathcal{B}_{N+1} is independent. For if $x = y_{N+1}$, then we can immediately solve for x^* as a linear combination of the other elements of \mathcal{B}_N . And if $x \neq y_{N+1}$, writing $x = a \cdot y_{N+1} + y$, where y is a linear combination of elements of $\mathcal{B}_N \setminus \{x, x^*\}$, we find that we can solve for x^* as a linear combination of the other elements of \mathcal{B}_N unless a = 0, but then x is a linear combination of the other elements of \mathcal{B}_N .

Thus the theorem is shown to hold for N+1 elements replaced given that it holds for N. By induction, the theorem holds in general.

Suppose that \mathcal{B} is a finite basis of n vectors and \mathcal{C} is an independent set at least as large as \mathcal{B} . By the theorem, we can replace all the vectors of \mathcal{B} by any n vectors from \mathcal{C} , the set of which must be a basis, and therefore a maximal independent set, so that \mathcal{C} actually contains just n elements. Hence, when a finite basis exists, no independent set can exceed it in size, and no basis can then be larger than any other basis. We therefore have the following important result.

Corollary 9 If a vector space has a finite basis, then each of its bases is finite, and each has the same number of elements. ■

A vector space is called **finite-dimensional** if it has a finite basis, and otherwise is called **infinite-dimensional**. The **dimension** of a vector space

is the number of elements in any basis for it if it is finite-dimensional, and ∞ otherwise. We write dim \mathcal{V} for the dimension of \mathcal{V} . If a finite-dimensional vector space has dimension n, then we say that it is n-dimensional. Over any field \mathcal{F} , \mathcal{F}^n is n-dimensional.

Exercise 1.13 In an n-dimensional vector space, any vector sequence with n+1 terms is dependent.

Exercise 1.14 Let the vector space V have the same finite dimension as its subspace U. Then U = V.

The corollary above has the following analog for infinite-dimensional spaces.

Theorem 10 All bases of an infinite-dimensional vector space have the same cardinality.

Proof: Let \mathcal{B} be an infinite basis for a vector space and let \mathcal{C} be another basis for the same space. For each $y \in \mathcal{C}$ let \mathcal{X}_y be the finite nonempty subset of \mathcal{B} such that y is the linear combination of its elements with nonzero scalar coefficients. Then each $x \in \mathcal{B}$ must appear in some \mathcal{X}_y , for if some $x \in \mathcal{B}$ appears in no \mathcal{X}_y , then that x could not be in the span of \mathcal{C} . Thus $\mathcal{B} = \bigcup_{y \in \mathcal{C}} \mathcal{X}_y$. Using $|\mathcal{S}|$ to denote cardinality of the set \mathcal{S} , we then have $|\mathcal{B}| = \left|\bigcup_{y \in \mathcal{C}} \mathcal{X}_y\right|$ and \mathcal{C} must be infinite for this to hold. Since \mathcal{C} is infinite, for each y we have $|\mathcal{X}_y| \leq |\mathcal{C}|$ and therefore $|\mathcal{B}| \leq |\mathcal{C} \times \mathcal{C}| = |\mathcal{C}|$, where the last equality is a well-known, but not so easily proved, result found in books on set theory and in online references such as the Wikipedia entry for Cardinal number. Similarly, we find $|\mathcal{C}| \leq |\mathcal{B}|$. The Schroeder-Bernstein theorem then gives $|\mathcal{B}| = |\mathcal{C}|$.

1.5 Sum and Direct Sum of Subspaces

By the **sum** of any number of subspaces is meant the span of their union. We use ordinary additive notation for subspace sums.

Exercise 1.15 For any sets of vectors, the sum of their spans is the span of their union.

Exercise 1.16 Each sum of subspaces is equal to the set of all finite sums of vectors selected from the various subspace summands, no two vector summands being selected from the same subspace summand.

The trivial subspace $\{0\}$ acts as a neutral element in subspace sums. Because of this, we will henceforth often write 0 for $\{0\}$, particularly when subspace sums are involved. By convention, the empty subspace sum is 0.

With $partial\ order \subset$, $meet \cap$, and join +, the subspaces of a given vector space form a $complete\ lattice$, which is modular, but generally not distributive.

Exercise 1.17 For subspaces, $(\mathcal{U}_1 \cap \mathcal{U}_3) + (\mathcal{U}_2 \cap \mathcal{U}_3) \subset (\mathcal{U}_1 + \mathcal{U}_2) \cap \mathcal{U}_3$, and (Modular Law) $\mathcal{U}_2 \triangleleft \mathcal{U}_3 \Rightarrow (\mathcal{U}_1 \cap \mathcal{U}_3) + \mathcal{U}_2 = (\mathcal{U}_1 + \mathcal{U}_2) \cap \mathcal{U}_3$.

Example 11 In \mathbb{R}^2 , let \mathcal{X} denote the x-axis, let \mathcal{Y} denote the y-axis, and let \mathcal{D} denote the line where y = x. Then $(\mathcal{X} + \mathcal{Y}) \cap \mathcal{D} = \mathcal{D}$ while $(\mathcal{X} \cap \mathcal{D}) + (\mathcal{Y} \cap \mathcal{D}) = 0$.

A sum of subspaces is called **direct** if the intersection of each summand with the sum of the other summands is 0. We often replace + with \oplus , or \sum with \bigoplus , to indicate that a subspace sum is direct.

The summands in a direct sum may be viewed as being independent, in some sense. Notice, though, that 0 may appear as a summand any number of times in a direct sum. Nonzero direct summands must be unique, however, and more generally, we have the following result.

Lemma 12 If a subspace sum is direct, then, for each summand \mathcal{U} , $\mathcal{U} \cap \Sigma = 0$, where Σ denotes the sum of any selection of the summands which does not include \mathcal{U} . In particular, two nonzero summands must be distinct.

Proof: $\Sigma \subset \overline{\Sigma}$ where $\overline{\Sigma}$ denotes the sum of all the summands with the exception of \mathcal{U} . By the definition of direct sum, $\mathcal{U} \cap \overline{\Sigma} = 0$, and hence $\mathcal{U} \cap \Sigma = 0$ as was to be shown.

This lemma leads immediately to the following useful result.

Theorem 13 If a subspace sum is direct, then the sum of any selection of the summands is direct. \blacksquare

There are a number of alternative characterizations of directness of sums of finitely many subspaces. (The interested reader will readily extend the first of these, and its corollary, to similar results for sums of infinitely many subspaces.)

Theorem 14 $u \in \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n \Leftrightarrow u = u_1 + \cdots + u_n$ for unique $u_1 \in \mathcal{U}_1, \ldots, u_n \in \mathcal{U}_n$.

Proof: Suppose that $u \in \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n$, and that $u = u_i + \overline{u}_i = v_i + \overline{v}_i$ where u_i and v_i are in \mathcal{U}_i and \overline{v}_i are in $\sum_{j \neq i} \mathcal{U}_j$. Then $w = u_i - v_i = \overline{v}_i - \overline{u}_i$ must be both in \mathcal{U}_i and in $\sum_{j \neq i} \mathcal{U}_j$ so that w = 0 and therefore $u_i = v_i$. Hence the representation of u as $u = u_1 + \cdots + u_n$, where $u_1 \in \mathcal{U}_1, \ldots, u_n \in \mathcal{U}_n$, is unique.

On the other hand, suppose that each $u \in \mathcal{U}_1 + \cdots + \mathcal{U}_n$ has the unique representation $u = u_1 + \cdots + u_n$ where $u_1 \in \mathcal{U}_1, \dots, u_n \in \mathcal{U}_n$. Let $u \in \mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j$. Then, since $u \in \mathcal{U}_i$, it has the representation $u = u_i$ for some $u_i \in \mathcal{U}_i$, and the uniqueness of the representation of u as $u = u_1 + \cdots + u_n$ where $u_1 \in \mathcal{U}_1, \dots, u_n \in \mathcal{U}_n$ then implies that $u_j = 0$ for all $j \neq i$. Also, since $u \in \sum_{j \neq i} \mathcal{U}_j$, it has the representation $u = \sum_{j \neq i} u_j$ for some $u_j \in \mathcal{U}_j$, and the uniqueness of the representation of u as $u = u_1 + \cdots + u_n$ where $u_1 \in \mathcal{U}_1, \dots, u_n \in \mathcal{U}_n$ implies that $u_i = 0$. Hence u = 0 and therefore for each $i, \mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j = 0$. The subspace sum therefore is direct.

Corollary 15 $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct if and only if 0 has a unique representation as $\sum_{1 \leq j \leq n} u_j$, where each $u_j \in \mathcal{U}_j$.

Proof: If $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct, then 0 has a unique representation of the form $\sum_{1 \leq j \leq n} u_j$, where each $u_j \in \mathcal{U}_j$, namely that one where each $u_j = 0$. On the other hand, suppose that 0 has that unique representation. Then given $\sum_{1 \leq j \leq n} u'_j = \sum_{1 \leq j \leq n} u''_j$, where each $u'_j \in \mathcal{U}_j$ and each $u''_j \in \mathcal{U}_j$, $0 = \sum_{1 \leq j \leq n} u_j$, where each $u_j = u'_j - u''_j \in \mathcal{U}_j$. Hence $u'_j - u''_j = 0$ for each j, and the subspace sum is direct.

Proposition 16 $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct if and only if both $\overline{\mathcal{U}}_n = \mathcal{U}_1 + \cdots + \mathcal{U}_{n-1}$ and $\overline{\mathcal{U}}_n + \mathcal{U}_n$ are direct, or, more informally, $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n = (\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_{n-1}) \oplus \mathcal{U}_n$.

Proof: Suppose that $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct. By Theorem 13, $\overline{\mathcal{U}}_n$ is direct, and the definition of direct sum immediately implies that $\overline{\mathcal{U}}_n + \mathcal{U}_n$ is direct.

On the other hand, suppose that both $\mathcal{U}_n = \mathcal{U}_1 + \cdots + \mathcal{U}_{n-1}$ and $\mathcal{U}_n + \mathcal{U}_n$ are direct. Then 0 has a unique representation of the form $u_1 + \cdots + u_{n-1}$, where $u_j \in \mathcal{U}_j$, and 0 has a unique representation of the form $\overline{u}_n + u_n$, where $\overline{u}_n \in \overline{\mathcal{U}}_n$ and $u_n \in \mathcal{U}_n$. Let 0 be represented in the form $u'_1 + \cdots + u'_n$,

where $u'_{j} \in \mathcal{U}_{j}$. Then $0 = \overline{u}'_{n} + u'_{n}$ where $\overline{u}'_{n} = u'_{1} + \cdots + u'_{n-1} \in \overline{\mathcal{U}}_{n}$. Hence $\overline{u}'_{n} = 0$ and $u'_{n} = 0$ by the corollary above, but then, by the same corollary, $u'_{1} = \cdots = u'_{n-1} = 0$. Using the corollary once more, $\mathcal{U}_{1} + \cdots + \mathcal{U}_{n}$ is proved to be direct. \blacksquare

Considering in turn $(\mathcal{U}_1 \oplus \mathcal{U}_2) \oplus \mathcal{U}_3$, $((\mathcal{U}_1 \oplus \mathcal{U}_2) \oplus \mathcal{U}_3) \oplus \mathcal{U}_4$, etc., we get the following result.

Corollary 17 $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct if and only if $(\mathcal{U}_1 + \cdots + \mathcal{U}_i) \cap \mathcal{U}_{i+1} = 0$ for $i = 1, \ldots, n-1$.

Exercise 1.18 Let $\Sigma_k = \mathcal{U}_{i_{k-1}+1} + \cdots + \mathcal{U}_{i_k}$ for $k = 1, \dots, K$, where $0 = i_0 < i_1 < \cdots < i_K = n$. Then $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct if and only if each Σ_k sum is direct and $\Sigma_1 + \cdots + \Sigma_n$ is direct.

Exercise 1.19 $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_6 = ((\mathcal{U}_1 \oplus (\mathcal{U}_2 \oplus \mathcal{U}_3)) \oplus (\mathcal{U}_4 \oplus \mathcal{U}_5)) \oplus \mathcal{U}_6$ in the fullest sense.

In the finite-dimensional case, additivity of dimension is characteristic.

Theorem 18 The finite-dimensional subspace sum $U_1 + \cdots + U_n$ is direct if and only if

$$\dim (\mathcal{U}_1 + \cdots + \mathcal{U}_n) = \dim \mathcal{U}_1 + \cdots + \dim \mathcal{U}_n.$$

Proof: Let each \mathcal{U}_j have the basis \mathcal{B}_j . It is clear that $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ spans the subspace sum.

Suppose now that $\mathcal{U}_1 + \cdots + \mathcal{U}_n$ is direct. $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is independent, and therefore a basis, because, applying the definition of direct sum, a dependency in it immediately implies one in one of the \mathcal{B}_j . And, of course, no element of any \mathcal{B}_i can be in any of the other \mathcal{B}_j , or it would be in the sum of their spans. Hence the \mathcal{B}_j are disjoint and dim $(\mathcal{U}_1 + \cdots + \mathcal{U}_n) = \dim \mathcal{U}_1 + \cdots + \dim \mathcal{U}_n$.

On the other hand, suppose that $\dim (\mathcal{U}_1 + \cdots + \mathcal{U}_n) = \dim \mathcal{U}_1 + \cdots + \dim \mathcal{U}_n$. Because a minimal spanning set is a basis, $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ must contain at least $\dim \mathcal{U}_1 + \cdots + \dim \mathcal{U}_n$ distinct elements, but clearly cannot contain more, and must therefore be a basis for the subspace sum. A nonzero element in $\mathcal{U}_i = \langle \mathcal{B}_i \rangle$ and simultaneously in the span of the other \mathcal{B}_j , would entail a dependency in the basis $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$, which of course is not possible. Hence no nonzero element of any \mathcal{U}_i can be in the sum of the other \mathcal{U}_j and therefore the subspace sum is direct. \blacksquare

Exercise 1.20 Let \mathcal{U} be any subspace of the vector space \mathcal{V} . As we already know, \mathcal{U} has a basis \mathcal{A} which is part of a basis \mathcal{B} for \mathcal{V} . Let $\overline{\mathcal{U}} = \langle \mathcal{B} \setminus \mathcal{A} \rangle$. Then $\mathcal{V} = \mathcal{U} \oplus \overline{\mathcal{U}}$.

Exercise 1.21 Let \mathcal{B} be a basis for \mathcal{V} . Then $\mathcal{V} = \bigoplus_{x \in \mathcal{B}} \langle \{x\} \rangle$.

1.6 Problems

- 1. Assuming only that + is a group operation, not necessarily that of an *Abelian group*, deduce from the rules for scalars operating on vectors that u + v = v + u for any vectors v and u.
- 2. Let \mathcal{T} be a subspace that does not contain the vectors u and v. Then

$$v \in \langle \mathcal{T} \cup \{u\} \rangle \Leftrightarrow u \in \langle \mathcal{T} \cup \{v\} \rangle$$
.

- 3. Let S and T be independent subsets of a vector space and suppose that S is finite and T is larger than S. Then T contains a vector x such that $S \cup \{x\}$ is independent. Deduce Corollary 9 from this.
- 4. Let \mathcal{B} and \mathcal{B}' be bases for the same vector space. Then for every $x \in \mathcal{B}$ there exists $x' \in \mathcal{B}'$ such that both $(\mathcal{B} \setminus \{x\}) \cup \{x'\}$ and $(\mathcal{B}' \setminus \{x'\}) \cup \{x\}$ are also bases.
- 5. Let \mathcal{W} and \mathcal{X} be finite-dimensional subspaces of a vector space. Then

$$\dim (\mathcal{W} \cap \mathcal{X}) \leq \dim \mathcal{X}$$
 and $\dim (\mathcal{W} + \mathcal{X}) \geq \dim \mathcal{W}$.

When does equality hold?

- 6. Let W and X be subspaces of a vector space V. Then V has a basis B such that each of W and X are spanned by subsets of B.
- 7. Instead of elements of a field \mathcal{F} , let the set of scalars be the integers \mathbb{Z} , so the result is not a vector space, but what is known as a \mathbb{Z} -module. \mathbb{Z}_3 , the integers modulo 3, is a \mathbb{Z} -module that has no linearly independent element. Thus \mathbb{Z}_3 has \emptyset as its only maximal linearly independent set, but $\langle \emptyset \rangle \neq \mathbb{Z}_3$. \mathbb{Z} itself is also a \mathbb{Z} -module, and has $\{2\}$ as a maximal linearly independent set, but $\langle \{2\} \rangle \neq \mathbb{Z}$. On the other hand, $\{1\}$ is also a maximal linearly independent set in \mathbb{Z} , and $\langle \{1\} \rangle = \mathbb{Z}$, so we

are prone to declare that $\{1\}$ is a basis for \mathbb{Z} . Every \mathbb{Z} -module, in fact, has a maximal linearly independent set, but the span of such a set may very well not be the whole \mathbb{Z} -module. We evidently should reject maximal linearly independent set as a definition of basis for \mathbb{Z} -modules, but linearly independent spanning set, equivalent for vector spaces, does give us a reasonable definition for those \mathbb{Z} -modules that are indeed spanned by an independent set. Let us adopt the latter as our definition for basis of a \mathbb{Z} -module. Then, is it possible for a \mathbb{Z} -module to have finite bases with differing numbers of elements? Also, is it then true that no finite \mathbb{Z} -module has a basis?

2 Fundamentals of Maps

2.1 Structure Preservation and Isomorphism

A structure-preserving function from a structured set to another with the same kind of structure is known in general as a **homomorphism**. The shorter term **map** is often used instead, as we shall do here. When the specific kind of structure is an issue, a qualifier may also be included, and so one speaks of group maps, ring maps, etc. We shall be concerned here mainly with vector space maps, so for us a map, unqualified, is a vector space map. (A vector space map is also commonly known as a **linear transformation**.)

For two vector spaces to be said to have the same kind of structure, we require them to be over the same field. Then for f to be a map between them, we require f(x+y) = f(x) + f(y) and $f(a \cdot x) = a \cdot f(x)$ for all vectors x, y and any scalar a, or, equivalently, that $f(a \cdot x + b \cdot y) = a \cdot f(x) + b \cdot f(y)$ for all vectors x, y and all scalars a, b. It is easy to show that a map sends 0 to 0, and a constant map must then send everything to 0. Under a map, the image of a subspace is always a subspace, and the inverse image of a subspace is always a subspace. The identity function on a vector space is a map. Maps which compose have a map as their composite.

Exercise 2.1 Let V and W be vector spaces over the same field and let S be a spanning set for V. Then any map $f: V \to W$ is already completely determined by its values on S.

The values of a map may be arbitrarily assigned on a minimal spanning set, but not on any larger spanning set.

Theorem 19 Let V and W be vector spaces over the same field and let B be a basis for V. Then given any function $f_0 : B \to W$, there is a unique map $f : V \to W$ such that f agrees with f_0 on B.

Proof: By the unique linear representation property of a basis, given $v \in \mathcal{V}$, there is a unique subset \mathcal{X} of \mathcal{B} and unique nonzero scalars a_x such that $v = \sum_{x \in \mathcal{X}} a_x \cdot x$. Because a map preserves linear combinations, any map f that agrees with f_0 on \mathcal{B} can only have the value $f(v) = \sum_{x \in \mathcal{X}} a_x \cdot f(x) = \sum_{x \in \mathcal{X}} a_x \cdot f_0(x)$. Setting $f(v) = \sum_{x \in \mathcal{X}} a_x \cdot f_0(x)$ does define a function $f: \mathcal{V} \to \mathcal{W}$ and this function f clearly agrees with f_0 on \mathcal{B} . Moreover, one immediately verifies that this f is a map. \blacksquare

It is standard terminology to refer to a function that is both one-to-one (injective) and onto (surjective) as bijective, or invertible. Each invertible $f: \mathcal{X} \to \mathcal{Y}$ has a (unique) $inverse\ f^{-1}$ such that $f^{-1} \circ f$ and $f \circ f^{-1}$ are the respective identity functions on \mathcal{X} and \mathcal{Y} , and an f that has such an f^{-1} is invertible. The composite of functions is bijective if and only if each individual function is bijective.

For some types of structure, the inverse of a bijective map need not always be a map itself. However, for vector spaces, inverting does always yield another map.

Theorem 20 The inverse of a bijective map is a map.

Proof:
$$f^{-1}(a \cdot v + b \cdot w) = a \cdot f^{-1}(v) + b \cdot f^{-1}(w)$$
 means precisely the same as $f(a \cdot f^{-1}(v) + b \cdot f^{-1}(w)) = a \cdot v + b \cdot w$, which is clearly true.

Corollary 21 A one-to-one map preserves independent sets, and conversely, a map that preserves independent sets is one-to-one.

Proof: A one-to-one map sends its domain bijectively onto its image and this image is a subspace of the codomain. Suppose a set in the one-to-one map's image is dependent. Then clearly the inverse image of this set is also dependent. An independent set therefore cannot be sent to a dependent set by a one-to-one map.

On the other hand, suppose that a map sends the distinct vectors u and v to the same image vector. Then it sends the nonzero vector v-u to 0, and hence it sends the independent set $\{v-u\}$ to the dependent set $\{0\}$.

Because their inverses are also maps, the bijective maps are the **isomorphisms** of vector spaces. If there is a bijective map from the vector space \mathcal{V} onto the vector space \mathcal{W} , we say that \mathcal{W} is **isomorphic** to \mathcal{V} . The notation $\mathcal{V} \cong \mathcal{W}$ is commonly used to indicate that \mathcal{V} and \mathcal{W} are isomorphic. Viewed as a relation between vector spaces, isomorphism is reflexive, symmetric and transitive, hence is an equivalence relation. If two spaces are isomorphic to each other we say that each is an **alias** of the other.

Theorem 22 Let two vector spaces be over the same field. Then they are isomorphic if and only if they have bases of the same cardinality.

Proof: Applying Theorem 19, the one-to-one correspondence between their bases extends to an isomorphism. On the other hand, an isomorphism restricts to a one-to-one correspondence between bases. \blacksquare

Corollary 23 Any n-dimensional vector space over the field \mathcal{F} is isomorphic to \mathcal{F}^n .

Corollary 24 (Fundamental Theorem of Finite-Dimensional Vector Spaces) Two finite-dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension. ■

The following well-known and often useful result may seem somewhat surprising on first encounter.

Theorem 25 A map of a finite-dimensional vector space into an alias of itself is one-to-one if and only if it is onto.

Proof: Suppose the map is one-to-one. Since one-to-one maps preserve independent sets, the image of a basis is an independent set with the same number of elements. The image of a basis must then be a maximal independent set, and hence a basis for the codomain, since the domain and codomain have equal dimension. Because a basis and its image are in one-to-one correspondence under the map, expressing elements in terms of these bases, it is clear that each element has a preimage element.

On the other hand, suppose the map is onto. The image of a basis is a spanning set for the codomain and must contain at least as many distinct elements as the dimension of the codomain. Since the domain and codomain have equal dimension, the image of a basis must in fact then be a minimal spanning set for the codomain, and therefore a basis for it. Expressing elements in terms of a basis and its image, we find that, due to unique representation, if the map sends u and v to the same element, then u = v.

For infinite-dimensional spaces, the preceding theorem fails to hold. It is a pleasant dividend of finite dimension, analogous to the result that a function between equinumerous finite sets is one-to-one if and only if it is onto.

2.2 Kernel, Level Sets and Quotient Space

The **kernel** of a map is the inverse image of $\{0\}$. The kernel of f is a subspace of the domain of f. A **level set** of a map is a nonempty inverse image of a singleton set. The kernel of a map, of course, is one of its level sets, the only

one that contains 0 and is a subspace. The level sets of a particular map make up a *partition* of the domain of the map. The following proposition internally characterizes the level sets as the *cosets* of the kernel.

Proposition 26 Let \mathcal{L} be a level set and \mathcal{K} be the kernel of the map f. Then for any v in \mathcal{L} , $\mathcal{L} = v + \mathcal{K} = \{v + x \mid x \in \mathcal{K}\}.$

Proof: Let $v \in \mathcal{L}$ and $x \in \mathcal{K}$. Then f(v+x) = f(v) + f(x) = f(v) + 0 so that $v+x \in \mathcal{L}$. Hence $v+\mathcal{K} \subset \mathcal{L}$. On the other hand, suppose that $u \in \mathcal{L}$ and write u=v+(u-v). Then since both u and v are in \mathcal{L} , f(u)=f(v), and hence f(u-v)=0, so that $u-v \in \mathcal{K}$. Hence $\mathcal{L} \subset v+\mathcal{K}$.

Corollary 27 A map is one-to-one if and only if its kernel is $\{0\}$.

Given any vector v in the domain of f, it must be in some level set, and now we know that this level set is $v + \mathcal{K}$, independent of the f which has domain \mathcal{V} and kernel \mathcal{K} .

Corollary 28 Maps with the same domain and the same kernel have identical level set families. For maps with the same domain, then, the kernel uniquely determines the level sets. ■

The level sets of a map f are in one-to-one correspondence with the elements of the image of the map. There is a very simple way in which the level sets can be made into a vector space isomorphic to the image of the map. Just use the one-to-one correspondence between the level sets and the image elements to give the image elements new labels according to the correspondence. Thus z gets the new label $f^{-1}(\{z\})$. To see what $f^{-1}(\{z\})+f^{-1}(\{w\})$ is, figure out what z+w is and then take $f^{-1}(\{z+w\})$, and similarly to see what $a \cdot f^{-1}(\{z\})$ is, figure out what $a \cdot z$ is and take $f^{-1}(\{a \cdot z\})$. All that has been done here is to give different names to the elements of the image of f. This level-set alias of the image of f is the vector space quotient (or quotient space) of the domain \mathcal{V} of f modulo the subspace $\mathcal{K} = \text{Kernel}(f)$ (which is the 0 of the space), and is denoted by \mathcal{V}/\mathcal{K} . The following result "internalizes" the operations in \mathcal{V}/\mathcal{K} .

Proposition 29 Let f be a map with domain \mathcal{V} and kernel \mathcal{K} . Let $\mathcal{L} = u + \mathcal{K}$ and $\mathcal{M} = v + \mathcal{K}$. Then in \mathcal{V}/\mathcal{K} , $\mathcal{L} + \mathcal{M} = (u + v) + \mathcal{K}$, and for any scalar a, $a \cdot \mathcal{L} = a \cdot u + \mathcal{K}$.

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Proof: \mathcal{L} = f^{-1}(\{f(u)\}) and \mathcal{M} = f^{-1}(\{f(v)\}). Hence \mathcal{L} + \mathcal{M} = f^{-1}(\{f(u) + f(v)\}) = f^{-1}(\{f(u + v)\}) = (u + v) + \mathcal{K}. Also, a \cdot \mathcal{L} = f^{-1}(\{a \cdot f(u)\}) = f^{-1}(\{f(a \cdot u)\}) = a \cdot u + \mathcal{K}.
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The following is immediate.

Corollary 30 V/K depends only on V and K, and not on the map f that has domain V and kernel K.

The next result tells us, for one thing, that kernel subspaces are not special, and hence the quotient space exists for any subspace \mathcal{K} . (In the vector space \mathcal{V} , the subspace $\overline{\mathcal{K}}$ is a **complementary subspace**, or **complement**, of the subspace \mathcal{K} if \mathcal{K} and $\overline{\mathcal{K}}$ have disjoint bases that together form a basis for \mathcal{V} , or what is the same, $\mathcal{V} = \mathcal{K} \bigoplus \overline{\mathcal{K}}$. Every subspace \mathcal{K} of \mathcal{V} has at least one complement $\overline{\mathcal{K}}$, because any basis \mathcal{A} of \mathcal{K} is contained in some basis \mathcal{B} of \mathcal{V} , and we may then take $\overline{\mathcal{K}} = \langle \mathcal{B} \setminus \mathcal{A} \rangle$.)

Proposition 31 Let K be a subspace of the vector space V and let \overline{K} be a complementary subspace of K. Then there is a map ϕ from V into itself which has K as its kernel and has \overline{K} as its image. Hence given any subspace K of V, the quotient space V/K exists and is an alias of any complementary subspace of K.

Proof: Let \mathcal{A} be a basis for \mathcal{K} , and let $\overline{\mathcal{A}}$ be a basis for $\overline{\mathcal{K}}$, so that $\mathcal{A} \cup \overline{\mathcal{A}}$ is a basis for \mathcal{V} . Define ϕ as the map of \mathcal{V} into itself which sends each element of \mathcal{A} to 0, and each element of $\overline{\mathcal{A}}$ to itself. Then ϕ has kernel \mathcal{K} and image $\overline{\mathcal{K}}$.

Supposing that K is the kernel of some map $f: \mathcal{V} \to \mathcal{W}$, the image of f is an alias of the quotient space, as is the image of the map ϕ of the proposition above. Hence any complementary subspace of the kernel of a map is an alias of the map's image. For maps with finite-dimensional domains, we then immediately deduce the following important and useful result. (The **rank** of a map is the dimension of its image, and the **nullity** of a map is the dimension of its kernel.)

Theorem 32 (Rank Plus Nullity Theorem) If the domain of a map has finite dimension, the sum of its rank and its nullity equals the dimension of its domain. ■

Exercise 2.2 Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be the map that sends (x, y, z) to (x - y, 0). Draw a picture that illustrates for this particular map the concepts of kernel and level set, and how complementary subspaces of the kernel are aliases of the image.

If we specify domain \mathcal{V} and kernel \mathcal{K} , we have the level sets turned into the vector space \mathcal{V}/\mathcal{K} as an alias of the image of any map with that domain and kernel. Thus \mathcal{V}/\mathcal{K} can be viewed as a generic image for maps with domain \mathcal{V} and kernel \mathcal{K} . To round out the picture, we now single out a map that sends \mathcal{V} onto \mathcal{V}/\mathcal{K} .

Proposition 33 The function $p: \mathcal{V} \to \mathcal{V}/\mathcal{K}$ which sends each $v \in \mathcal{V}$ to $v + \mathcal{K}$ is a map with kernel \mathcal{K} .

Proof: Supposing, as we may, that \mathcal{K} is the kernel of some map $f: \mathcal{V} \to \mathcal{W}$, let the map $f^{\flat}: \mathcal{V} \to \operatorname{Image}(f)$ be defined by $f^{\flat}(v) = f(v)$ for all $v \in \mathcal{V}$. Let $\Theta: \operatorname{Image}(f) \to \mathcal{V}/\mathcal{K}$ be the isomorphism from the image of f to \mathcal{V}/\mathcal{K} obtained through the correspondence of image elements with their level sets. Notice that Θ sends w = f(v) to $v + \mathcal{K}$. Then $p = \Theta \circ f^{\flat}$ and is obviously a map with kernel \mathcal{K} .

We call p the **natural projection**. It serves as a generic map for obtaining the generic image. For any map $f: \mathcal{V} \to \mathcal{W}$ with kernel \mathcal{K} , we always have the composition

$$\mathcal{V} \to \mathcal{V}/\mathcal{K} \longleftrightarrow \operatorname{Image}(f) \hookrightarrow \mathcal{W}$$

where $V \to V/K$ is generic, denoting the natural projection p, and the remainder is the nongeneric specialization to f via an isomorphism and an inclusion map. Our next result details how the generic map p in fact serves in a more general way as a universal factor of maps with kernel containing K.

Theorem 34 Let $p: \mathcal{V} \to \mathcal{V}/\mathcal{K}$ be the natural projection and let $f: \mathcal{V} \to \mathcal{W}$. If $\mathcal{K} \subset \text{Kernel}(f)$ then there is a unique **induced map** $f_{\mathcal{V}/\mathcal{K}}: \mathcal{V}/\mathcal{K} \to \mathcal{W}$ such that $f_{\mathcal{V}/\mathcal{K}} \circ p = f$.

Proof: The prescription $f_{\mathcal{V}/\mathcal{K}} \circ p = f$ determines exactly what $f_{\mathcal{V}/\mathcal{K}}$ must do: for each $v \in \mathcal{V}$, $f_{\mathcal{V}/\mathcal{K}}(p(v)) = f_{\mathcal{V}/\mathcal{K}}(v + \mathcal{K}) = f(v)$. For this to unambiguously define the value of $f_{\mathcal{V}/\mathcal{K}}$ at each element of \mathcal{V}/\mathcal{K} , it must be the

case that if $u + \mathcal{K} = v + \mathcal{K}$, then f(u) = f(v). But this is true because $\mathcal{K} \subset \text{Kernel}(f)$ implies that f(k) = 0 for each $k \in \mathcal{K}$. The unique function $f_{\mathcal{V}/\mathcal{K}}$ so determined is readily shown to be a map.

Exercise 2.3 Referring to the theorem above, $f_{\mathcal{V}/\mathcal{K}}$ is one-to-one if and only if $\operatorname{Kernel}(f) = \mathcal{K}$, and $f_{\mathcal{V}/\mathcal{K}}$ is onto if and only if f is onto.

The rank of the natural map $p: \mathcal{V} \to \mathcal{V}/\mathcal{K}$ is the dimension of \mathcal{V}/\mathcal{K} , also known as the **codimension** of the subspace \mathcal{K} and denoted by codim \mathcal{K} . Thus the Rank Plus Nullity Theorem may also be expressed as

$$\dim \mathcal{K} + \operatorname{codim} \mathcal{K} = \dim \mathcal{V}$$

when dim \mathcal{V} is finite.

Exercise 2.4 Let K be a subspace of finite codimension in the infinite-dimensional vector space V. Then $K \cong V$.

2.3 Short Exact Sequences

There is a special type of map sequence that is a way to view a quotient. The map sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is said to be **exact** at \mathcal{Y} if the image of the map going into \mathcal{Y} is exactly the kernel of the map coming out of \mathcal{Y} . A short **exact sequence** is a map sequence of the form $0 \to \mathcal{K} \to \mathcal{V} \to \mathcal{W} \to 0$ which is exact at \mathcal{K} , \mathcal{V} , and \mathcal{W} . Notice that here the map $\mathcal{K} \to \mathcal{V}$ is one-toone, and the map $\mathcal{V} \to \mathcal{W}$ is onto, in virtue of the exactness requirements and the restricted nature of maps to or from $0 = \{0\}$. Hence if $\mathcal{V} \to \mathcal{W}$ corresponds to the function f, then it is easy to see that \mathcal{K} is an alias of Kernel(f), and W is an alias of V/Kernel(f). Thus the short exact sequence captures the idea of quotient modulo any subspace. It is true that with the same \mathcal{K} , \mathcal{V} , and \mathcal{W} , there are maps such that the original sequence with all its arrows reversed is also a short exact sequence, one complementary to the original, it might be said. This is due to the fact that K and W are always aliases of complementary subspaces of \mathcal{V} , which is something special that vector space structure makes possible. Thus it is said that, for vector spaces, a short exact sequence always splits, as illustrated by the short exact sequence $0 \to \mathcal{K} \to \mathcal{K} \oplus \overline{\mathcal{K}} \to \overline{\mathcal{K}} \to 0$.

An example of the use of the exact sequence view of quotients stems from considering the following diagram where the rows and columns are short exact sequences. Each second map in a row or column is assumed to be an inclusion, and each third map is assumed to be a natural projection.

The sequence $0 \to \mathcal{H} \to \mathcal{V} \to \mathcal{V}/\mathcal{K} \to (\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K}) \to 0$ is a subdiagram. If we compose the pair of onto maps of $\mathcal{V} \to \mathcal{V}/\mathcal{K} \to (\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K})$ to yield the composite onto map $\mathcal{V} \to (\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K})$, we then have the sequence $0 \to \mathcal{H} \to \mathcal{V} \to (\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K}) \to 0$ which is exact at \mathcal{H} and at $(\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K})$, and would also be exact at \mathcal{V} if \mathcal{H} were the kernel of the composite. But it is precisely the $h \in \mathcal{H}$ that map to the $h + \mathcal{K} \in \mathcal{H}/\mathcal{K} \subset \mathcal{V}/\mathcal{K}$ which then are precisely the elements that map to \mathcal{H}/\mathcal{K} which is the 0 of $(\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K})$. Thus we have obtained the following isomorphism theorem.

Theorem 35 Let K be a subspace of \mathcal{H} , and let \mathcal{H} be a subspace of \mathcal{V} . Then \mathcal{V}/\mathcal{H} is isomorphic to $(\mathcal{V}/\mathcal{K})/(\mathcal{H}/\mathcal{K})$.

Exercise 2.5 Let V be \mathbb{R}^3 , let \mathcal{H} be the (x,y)-plane, and let \mathcal{K} be the x-axis. Interpret the theorem above for this case.

Now consider the diagram below where \mathcal{X} and \mathcal{Y} are subspaces and again the rows and columns are short exact sequences and the second maps are inclusions while the third maps are natural projections.

As a subdiagram we have the sequence $0 \to \mathcal{X} \cap \mathcal{Y} \to \mathcal{X} \to \mathcal{X} + \mathcal{Y} \to (\mathcal{X} + \mathcal{Y})/\mathcal{Y} \to 0$. Replacing the sequence $\mathcal{X} \to \mathcal{X} + \mathcal{Y} \to (\mathcal{X} + \mathcal{Y})/\mathcal{Y}$ with its composite, the result would be a short exact sequence if the composite $\mathcal{X} \to (\mathcal{X} + \mathcal{Y})/\mathcal{Y}$ were an onto map with kernel $\mathcal{X} \cap \mathcal{Y}$.

To see if the composite is onto, we must see if each element of $(\mathcal{X} + \mathcal{Y})/\mathcal{Y}$ is the image of some element of \mathcal{X} . Now each element of $(\mathcal{X} + \mathcal{Y})/\mathcal{Y}$ is the image of some element $w \in \mathcal{X} + \mathcal{Y}$ where w = u + v for some $u \in \mathcal{X}$ and some $v \in \mathcal{Y}$. But clearly the element u by itself has the same image modulo \mathcal{Y} . Hence the composite is onto. Also, the elements of \mathcal{X} that map to the 0 of $(\mathcal{X} + \mathcal{Y})/\mathcal{Y}$, namely \mathcal{Y} , are precisely those that then are also in \mathcal{Y} . Hence the kernel of the composite is $\mathcal{X} \cap \mathcal{Y}$.

Thus we have obtained another isomorphism theorem and an immediate corollary.

Theorem 36 Let \mathcal{X} and \mathcal{Y} be any subspaces of some vector space. Then $\mathcal{X}/(\mathcal{X}\cap\mathcal{Y})$ is isomorphic to $(\mathcal{X}+\mathcal{Y})/\mathcal{Y}$.

Corollary 37 (Grassmann's Relation) Let X and Y be finite-dimensional subspaces of a vector space. Then

$$\dim \mathcal{X} + \dim \mathcal{Y} = \dim(\mathcal{X} + \mathcal{Y}) + \dim(\mathcal{X} \cap \mathcal{Y}). : EndProof$$

Exercise 2.6 Two 2-dimensional subspaces of \mathbb{R}^3 are either identical or intersect in a 1-dimensional subspace.

Exercise 2.7 Let \mathcal{N} , \mathcal{N}' , \mathcal{P} , and \mathcal{P}' be subspaces of the same vector space and let $\mathcal{N} \subset \mathcal{P}$ and $\mathcal{N}' \subset \mathcal{P}'$. Set $\mathcal{X} = \mathcal{P} \cap \mathcal{P}'$ and $\mathcal{Y} = (\mathcal{P} \cap \mathcal{N}') + \mathcal{N}$. Verify that $\mathcal{X} \cap \mathcal{Y} = (\mathcal{P}' \cap \mathcal{N}) + (\mathcal{P} \cap \mathcal{N}')$ and $\mathcal{X} + \mathcal{Y} = (\mathcal{P} \cap \mathcal{P}') + \mathcal{N}$, so that

$$\frac{\left(\mathcal{P}^{'}\cap\mathcal{P}\right)+\mathcal{N}^{'}}{\left(\mathcal{P}^{'}\cap\mathcal{N}\right)+\mathcal{N}^{'}}\cong\frac{\mathcal{P}\cap\mathcal{P}^{'}}{\left(\mathcal{P}^{'}\cap\mathcal{N}\right)+\left(\mathcal{P}\cap\mathcal{N}^{'}\right)}\cong\frac{\left(\mathcal{P}\cap\mathcal{P}^{'}\right)+\mathcal{N}}{\left(\mathcal{P}\cap\mathcal{N}^{'}\right)+\mathcal{N}}\ .$$

2.4 Projections and Reflections on $V = W \oplus X$

Relative to the decomposition of the vector space \mathcal{V} by a pair \mathcal{W}, \mathcal{X} of direct summands, we may define some noteworthy self-maps on \mathcal{V} . Expressing the general vector $v \in \mathcal{V} = \mathcal{W} \oplus \mathcal{X}$ uniquely as v = w + x where $w \in \mathcal{W}$ and $x \in \mathcal{X}$, two types of such are defined by

$$P_{\mathcal{W}|\mathcal{X}}(v) = w$$
 and $R_{\mathcal{W}|\mathcal{X}}(v) = w - x$.

That these really are maps is easy to establish. We call $P_{W|\mathcal{X}}$ the **projection** onto W along \mathcal{X} and we call $R_{W|\mathcal{X}}$ the **reflection in** W along \mathcal{X} . (The function ϕ in the proof of Proposition 31 was the projection onto $\overline{\mathcal{K}}$ along \mathcal{K} , so we have already employed the projection type to advantage.) Denoting the identity map on \mathcal{V} by I, we have

$$P_{\mathcal{X}|\mathcal{W}} = I - P_{\mathcal{W}|\mathcal{X}}$$

and

$$R_{\mathcal{W}|\mathcal{X}} = P_{\mathcal{W}|\mathcal{X}} - P_{\mathcal{X}|\mathcal{W}} = I - 2P_{\mathcal{X}|\mathcal{W}} = 2P_{\mathcal{W}|\mathcal{X}} - I.$$

It bears mention that a given W generally has many different complements, and if \mathcal{X} and \mathcal{Y} are two such, $P_{W|\mathcal{X}}$ and $P_{W|\mathcal{Y}}$ will generally differ.

The image of $P_{W|\mathcal{X}}$ is W and its kernel is \mathcal{X} . Thus $P_{W|\mathcal{X}}$ is a self-map with the special property that its image and its kernel are complements. The image and kernel of $R_{W|\mathcal{X}}$ are also complements, but trivially, as the kernel of $R_{W|\mathcal{X}}$ is $0 = \{0\}$. The kernel of $R_{W|\mathcal{X}}$ is 0 because w and x are equal only if they are both 0 since $W \cap \mathcal{X} = 0$ from the definition of direct sum.

A double application of $P_{W|\mathcal{X}}$ has the same effect as a single application: $P_{W|\mathcal{X}}^2 = P_{W|\mathcal{X}} \circ P_{W|\mathcal{X}} = 0$ (the constant map on \mathcal{V} that sends everything to 0). We also find that $P_{W|\mathcal{X}}^2 = P_{W|\mathcal{X}}$ implies that $R_{W|\mathcal{X}}^2 = I (R_{W|\mathcal{X}} \text{ is } involutary)$. An idempotent map always turns out to be some $P_{W|\mathcal{X}}$, and an involutary map is almost always some $R_{W|\mathcal{X}}$, as we now explain.

Proposition 38 Let $P: \mathcal{V} \to \mathcal{V}$ be a map such that $P^2 = P$. Then P is the projection onto Image (P) along Kernel (P).

Proof: Let w be in Image (P), say w = P(v). Then

$$w = P^{2}(v) = P(P(v)) = P(w)$$

so that if w is also in Kernel (P), it must be that w = 0. Hence

$$\operatorname{Image}(P) \cap \operatorname{Kernel}(P) = 0.$$

Let Q = I - P so that P(v) + Q(v) = I(v) = v and thus Image (P) + Image(Q) = V. But Image (Q) = Kernel(P) since if x = Q(v) = I(v) - P(v), then $P(x) = P(v) - P^2(v) = 0$. Therefore

$$\mathcal{V} = \operatorname{Image}(P) \oplus \operatorname{Kernel}(P)$$
.

Now if v = w + x, $w \in \text{Image}(P)$, $x \in \text{Kernel}(P)$, then P(v) = P(w + x) = P(w) + P(x) = w + 0 = w, and P is indeed the projection onto Image (P) along Kernel (P).

When \mathcal{V} is over a field in which 1+1=0, the only possibility for $R_{\mathcal{W}|\mathcal{X}}$ is the identity I. However, when \mathcal{V} is over such a field there can be an involutary map on \mathcal{V} which is not the identity, for example the map of the following exercise.

Exercise 2.8 Over the 2-element field $\mathcal{F} = \{0,1\}$ (wherein 1+1=0), let \mathcal{V} be the vector space \mathcal{F}^2 of 2-tuples of elements of \mathcal{F} . Let F be the map on \mathcal{V} that interchanges (0,1) and (1,0). Then $F \circ F$ is the identity map.

The F of the exercise can reasonably be called a reflection, but it is not some $R_{\mathcal{W}|\mathcal{X}}$. If we say that any involutary map is a reflection, then reflections of the form $R_{\mathcal{W}|\mathcal{X}}$ do not quite cover all the possibilities. However, when $1+1\neq 0$, they do. (For any function $f:\mathcal{V}\to\mathcal{V}$, we denote by Fix (f) the set of all elements in \mathcal{V} that are fixed by f, i. e., Fix $(f) = \{v \in \mathcal{V} \mid f(v) = v\}$.)

Exercise 2.9 Let V be over a field such that $1 + 1 \neq 0$. Let $R : V \to V$ be a map such that $R^2 = I$. Then R = 2P - I, where $P = \frac{1}{2}(R + I)$ is the projection onto Fix(R) along Fix(-R).

2.5 Problems

1. Let $f: S \to T$ be a function. Then for all subsets A, B of S,

$$f(A \cap B) \subset f(A) \cap f(B)$$
,

but

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A, B of S if and only if f is one-to-one. What about for \cup instead of \cap ? And what about for \cap and for \cup when f is replaced by f^{-1} ?

2. (Enhancing Theorem 19 for nontrivial codomains) Let \mathcal{B} be a subset of the vector space \mathcal{V} and let $\mathcal{W} \neq \{0\}$ be a vector space over the same field. Then \mathcal{B} is independent if and only if every function $f_0 : \mathcal{B} \to \mathcal{W}$ extends to at least one map $f : \mathcal{V} \to \mathcal{W}$, and \mathcal{B} spans \mathcal{V} if and only if every function

 $f_0: \mathcal{B} \to \mathcal{W}$ extends to at most one map $f: \mathcal{V} \to \mathcal{W}$. Hence \mathcal{B} is a basis for \mathcal{V} if and only if every function $f_0: \mathcal{B} \to \mathcal{W}$ extends uniquely to a map $f: \mathcal{V} \to \mathcal{W}$.

- 3. Let \mathcal{B} be a basis for the vector space \mathcal{V} and let $f: \mathcal{V} \to \mathcal{W}$ be a map. Then f is one-to-one if and only if $f(\mathcal{B})$ is an independent set, and f is onto if and only if $f(\mathcal{B})$ spans \mathcal{W} .
 - 4. Deduce Theorem 25 as a corollary of the Rank Plus Nullity Theorem.
- 5. Let \mathcal{V} be a finite-dimensional vector space of dimension n over the finite field of q elements. Then the number of vectors in \mathcal{V} is q^n , the number of nonzero vectors that are not in a given m-dimensional subspace is

$$(q^{n}-1)-(q^{m}-1)=(q^{n}-q^{m}),$$

and the number of different basis sets that can be extracted from $\mathcal V$ is

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}{n!}$$

since there are

$$(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$$

different sequences of n distinct basis vectors that can be drawn from \mathcal{V} .

- 6. (Correspondence Theorem) Let $\mathcal{U} \triangleleft \mathcal{V}$. The natural projection $p : \mathcal{V} \rightarrow \mathcal{V}/\mathcal{U}$ puts the subspaces of \mathcal{V} that contain \mathcal{U} in one-to-one correspondence with the subspaces of \mathcal{V}/\mathcal{U} , and every subspace of \mathcal{V}/\mathcal{U} is of the form \mathcal{W}/\mathcal{U} for some subspace \mathcal{W} that contains \mathcal{U} .
- 7. $P_{\mathcal{W}_1|\mathcal{X}_1} + P_{\mathcal{W}_2|\mathcal{X}_2} = P_{\mathcal{W}|\mathcal{X}}$ if and only if $P_{\mathcal{W}_1|\mathcal{X}_1} \circ P_{\mathcal{W}_2|\mathcal{X}_2} = P_{\mathcal{W}_2|\mathcal{X}_2} \circ P_{\mathcal{W}_1|\mathcal{X}_1} = 0$, in which case $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_2$. What happens when $P_{\mathcal{W}_1|\mathcal{X}_1} P_{\mathcal{W}_2|\mathcal{X}_2} = P_{\mathcal{W}|\mathcal{X}}$ is considered instead?

3 More on Maps and Structure

3.1 Function Spaces and Map Spaces

The set $\mathcal{W}^{\mathcal{D}}$ of all functions from a set \mathcal{D} into the vector space \mathcal{W} may be made into a vector space by defining the operations in a pointwise fashion. That is, suppose that f and g are in $\mathcal{W}^{\mathcal{D}}$, and define f + g by (f + g)(x) = f(x) + g(x), and for any scalar a define $a \cdot f$ by $(a \cdot f)(x) = a \cdot (f(x))$. This makes $\mathcal{W}^{\mathcal{D}}$ into a vector space, and it is this vector space that we mean whenever we refer to $\mathcal{W}^{\mathcal{D}}$ as a **function space**. Considering n-tuples as functions with domain $\{1, \ldots, n\}$ and the field \mathcal{F} as a vector space over itself, we see that the familiar \mathcal{F}^n is an example of a function space.

Subspaces of a function space give us more examples of vector spaces. For instance, consider the set $\{\mathcal{V} \to \mathcal{W}\}$ of all maps from the vector space \mathcal{V} into the vector space \mathcal{W} . It is easy to verify that it is a subspace of the function space $\mathcal{W}^{\mathcal{V}}$. It is this subspace that we mean when we refer to $\{\mathcal{V} \to \mathcal{W}\}$ as a **map space**.

3.2 Dual of a Vector Space, Dual of a Basis

By a **linear functional** on a vector space \mathcal{V} over a field \mathcal{F} is meant an element of the map space $\{\mathcal{V} \to \mathcal{F}^1\}$. (Since \mathcal{F}^1 is just \mathcal{F} viewed as a vector space over itself, a linear functional is also commonly described as a map into \mathcal{F} .) For a vector space \mathcal{V} over a field \mathcal{F} , the map space $\{\mathcal{V} \to \mathcal{F}^1\}$ will be called the **dual** of \mathcal{V} and will be denoted by \mathcal{V}^{\top} .

Let the vector space \mathcal{V} have the basis \mathcal{B} . For each $x \in \mathcal{B}$, let x^{\top} be the linear functional on \mathcal{V} that sends x to 1 and sends all other elements of \mathcal{B} to 0. We call x^{\top} the **coordinate function** corresponding to the basis vector x. The set $\mathcal{B}^{\#}$ of all such coordinate functions is an independent set. For if for some finite $\mathcal{X} \subset \mathcal{B}$ we have $\sum_{x \in \mathcal{X}} a_x \cdot x^{\top}(v) = 0$ for all $v \in \mathcal{V}$, then for any $v \in \mathcal{X}$ we find that $a_v = 0$ so there can be no dependency in $\mathcal{B}^{\#}$.

When the elements of $\mathcal{B}^{\#}$ span \mathcal{V}^{\top} , we say that \mathcal{B} has $\mathcal{B}^{\#}$ as its **dual** and we may then replace $\mathcal{B}^{\#}$ with the more descriptive notation \mathcal{B}^{\top} . When \mathcal{V} is finite-dimensional, $\mathcal{B}^{\#}$ does span \mathcal{V}^{\top} , we claim. For the general linear functional f is defined by prescribing its value on each element of \mathcal{B} , and given any $v \in \mathcal{B}$ we have $f(v) = \sum_{x \in \mathcal{B}} f(x) \cdot x^{\top}(v)$. Therefore any linear functional on \mathcal{V} may be expressed as the linear combination $f = \sum_{x \in \mathcal{B}} f(x) \cdot x^{\top}$, so that $\mathcal{B}^{\#}$ spans \mathcal{V}^{\top} as we claimed.

We thus have the following result.

Theorem 39 Let V be a finite-dimensional vector space. Then any basis of V has a dual, and therefore V^{\top} has the same dimension as V.

When \mathcal{V} is infinite-dimensional, $\mathcal{B}^{\#}$ need not span \mathcal{V}^{\top} .

Example 40 Let \mathbb{N} denote the natural numbers $\{0,1,\ldots\}$. Let \mathcal{V} be the subspace of the function space $\mathbb{R}^{\mathbb{N}}$ consisting of only those real sequences a_0, a_1, \ldots that are ultimately zero, i. e., for which there exists a smallest $N \in \mathbb{N}$ (called the length of the ultimately zero sequence), such that $a_n = 0$ for all $n \geq N$. Then $\phi : \mathcal{V} \to \mathbb{R}$ defined by $\phi(a_0, a_1, \ldots) = \sum_{n \in \mathbb{N}} \alpha_n a_n$ is a linear functional given any (not necessarily ultimately zero) sequence of real coefficients $\alpha_0, \alpha_1, \ldots$ (Given $\alpha_0, \alpha_1, \ldots$ that is not ultimately zero, it cannot be replaced by an ultimately zero sequence that produces the same functional, since there would always be sequences a_0, a_1, \ldots of greater length for which the results would differ.) The $\mathcal{B}^\#$ derived from the basis $\mathcal{B} = \{(1,0,0,\ldots),(0,1,0,0,\ldots),\ldots\}$ for \mathcal{V} does not span \mathcal{V}^\top because $\langle \mathcal{B}^\# \rangle$ only contains elements corresponding to sequences $\alpha_0, \alpha_1, \ldots$ that are ultimately zero. Also note that by Theorem 19, each element of \mathcal{V}^\top is determined by an unrestricted choice of value at each and every element of \mathcal{B} .

Fixing an element $v \in \mathcal{V}$, the scalar-valued function F_v defined on \mathcal{V}^{\top} by $F_v(f) = f(v)$ is a linear functional on \mathcal{V}^{\top} , i. e., F_v is an element of $\mathcal{V}^{\top\top} = (\mathcal{V}^{\top})^{\top}$. The association of each such $v \in \mathcal{V}$ with the corresponding $F_v \in \mathcal{V}^{\top\top}$ gives a map Θ from \mathcal{V} into $\mathcal{V}^{\top\top}$. Θ is one-to-one, we claim. For suppose that v and w are in \mathcal{V} and that $F_v = F_w$ so that for all $f \in \mathcal{V}^{\top}$, f(v) = f(w). Then f(v - w) = 0 for every $f \in \mathcal{V}^{\top}$. Let \mathcal{B} be a basis for \mathcal{V} , and for each $x \in \mathcal{B}$ let x^{\top} be the corresponding coordinate function. Let \mathcal{X} be a finite subset of \mathcal{B} such that $v - w = \sum_{x \in \mathcal{X}} a_x \cdot x$. Applying x^{\top} to v - w for each $x \in \mathcal{X}$, we conclude that there is no nonzero a_x . Hence v = w, and Θ is one-to-one as claimed.

Thus it has been shown that each vector $v \in \mathcal{V}$ defines a unique element $F_v \in \mathcal{V}^{\top \top}$. But the question still remains as to whether Θ is onto, that is whether every $F \in \mathcal{V}^{\top \top}$ is an F_v for some $v \in \mathcal{V}$. We claim that Θ is onto if \mathcal{V} is finite-dimensional. This is because if we have bases \mathcal{B} and \mathcal{B}^{\top} as above and we form the basis $\mathcal{B}^{\top \top} = \{x_1^{\top \top}, \dots, x_n^{\top \top}\}$ dual to \mathcal{B}^{\top} , we readily verify that $x_i^{\top \top} = F_{x_i}$ for each i. Thus for $F = a_1 \cdot x_1^{\top \top} + \dots + a_n \cdot x_n^{\top \top}$, and $v = a_1 \cdot x_1 + \dots + a_n \cdot x_n$, we have $F = F_v$.

We have proved the following.

Theorem 41 The map $\Theta: \mathcal{V} \to \mathcal{V}^{\top \top}$ which sends $v \in \mathcal{V}$ to the linear functional F_v on \mathcal{V}^{\top} defined by $F_v(f) = f(v)$ is always one-to-one, and moreover is onto if \mathcal{V} is finite-dimensional.

 Θ does not depend on any choice of basis, so \mathcal{V} and $\Theta(\mathcal{V}) \subset \mathcal{V}^{\top\top}$ are isomorphic in a "natural" basis-invariant manner. We say that Θ naturally embeds \mathcal{V} in $\mathcal{V}^{\top\top}$ and we call Θ the natural injection of \mathcal{V} into $\mathcal{V}^{\top\top}$, or, when it is onto, the natural correspondence between \mathcal{V} and $\mathcal{V}^{\top\top}$. Independent of basis choices, a vector in \mathcal{V} can be considered to act directly on \mathcal{V}^{\top} as a linear functional. If one writes f(v), where $f \in \mathcal{V}^{\top}$ and $v \in \mathcal{V}$, either of f or v can be the variable. When f is the variable, then v is really playing the rôle of a linear functional on \mathcal{V}^{\top} , i. e., an element of $\mathcal{V}^{\top\top}$. One often sees notation such as $\langle f, v \rangle$ used for f(v) in order to more clearly indicate that f and v are to viewed as being on similar footing, with either being a functional of the other. Thus, as convenience dictates, we may identify \mathcal{V} with $\Theta(\mathcal{V}) \subset \mathcal{V}^{\top\top}$, and in particular, we agree always to do so whenever Θ is onto.

3.3 Annihilators

Let S be a subset of the vector space V. By the **annihilator** S^0 of S is meant the subset of V^{\top} which contains all the linear functionals f such that f(s) = 0 for all $s \in S$. It is easy to see that S^0 is actually a subspace of V^{\top} and that $S^0 = \langle S \rangle^0$. Obviously, $\{0\}^0 = V^{\top}$. Also, since a linear functional that is not the zero functional, i. e., which is not identically zero, must have at least one nonzero value, $V^0 = \{0\}$.

Exercise 3.1 Because of the natural injection of \mathcal{V} into $\mathcal{V}^{\top \top}$, the only $x \in \mathcal{V}$ such that f(x) = 0 for all $f \in \mathcal{V}^{\top}$ is x = 0.

When $\mathcal V$ is finite-dimensional, the dimension of a subspace determines that of its annihilator.

Theorem 42 Let V be an n-dimensional vector space and let $U \triangleleft V$ have dimension m. Then U^0 has dimension n-m.

Proof: Let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis for \mathcal{V} such that $\{x_1, \dots, x_m\}$ is a basis for \mathcal{U} . We claim that $\{x_{m+1}^\top, \dots, x_n^\top\} \subset \mathcal{B}^\top$ spans \mathcal{U}^0 . Clearly, $\langle \{x_{m+1}^\top, \dots, x_n^\top\} \rangle \subset \mathcal{U}^0$. On the other hand, if $f = b_1 \cdot x_1^\top + \dots + b_n \cdot x_n^\top \in \mathcal{U}^0$

then f(x) = 0 for each $x \in \{x_1, \dots, x_m\}$ so that $b_1 = \dots = b_m = 0$ and therefore $\mathcal{U}^0 \subset \langle \{x_{m+1}^\top, \dots, x_n^\top\} \rangle$.

Exercise 3.2 Let V be a finite-dimensional vector space. For any $U \triangleleft V$, $(U^0)^0 = U$, remembering that we identify V with $V^{\top \top}$.

3.4 Dual of a Map

Let $f: \mathcal{V} \to \mathcal{W}$ be a map. Then for $\varphi \in \mathcal{W}^{\top}$, the prescription $f^{\top}(\varphi) = \varphi \circ f$, or $(f^{\top}(\varphi))(v) = \varphi(f(v))$ for all $v \in \mathcal{V}$, defines a function $f^{\top}: \mathcal{W}^{\top} \to \mathcal{V}^{\top}$. It is easy to show that f^{\top} is a map. f^{\top} is called the **dual** of f. The dual of an identity map is an identity map. The dual of the composite map $f \circ g$ is $g^{\top} \circ f^{\top}$. Thus, if f is a bijective map, the dual of its inverse is the inverse of its dual.

Exercise 3.3 Let \mathcal{V} and \mathcal{W} be finite-dimensional vector spaces with respective bases \mathcal{B} and \mathcal{C} . Let the map $f: \mathcal{V} \to \mathcal{W}$ satisfy $f(x) = \sum_{y \in \mathcal{C}} a_{y,x} \cdot y$ for each $x \in \mathcal{B}$. Then f^{\top} satisfies $f^{\top}(y^{\top}) = \sum_{x \in \mathcal{B}} a_{y,x} \cdot x^{\top}$ for each $y \in \mathcal{C}$ (where $x^{\top} \in \mathcal{B}^{\top}$ and $y^{\top} \in \mathcal{C}^{\top}$ are coordinate functions corresponding respectively to $x \in \mathcal{B}$ and $y \in \mathcal{C}$, of course.)

The kernel of f^{\top} is found to have a special property.

Theorem 43 The kernel of f^{\top} is the annihilator of the image of the map $f: \mathcal{V} \to \mathcal{W}$.

Proof: φ is in the kernel of $f^{\top} \Leftrightarrow (f^{\top}(\varphi))(v) = 0$ for all $v \in \mathcal{V} \Leftrightarrow \varphi(f(v)) = 0$ for all $v \in \mathcal{V} \Leftrightarrow \varphi \in (f(\mathcal{V}))^0$.

When the codomain of f has finite dimension m, say, then the image of f has finite dimension r, say. Hence, the kernel of f^{\top} (which is the annihilator of the image of f) must have dimension m-r. By the Rank Plus Nullity Theorem, the rank of f^{\top} must then be r. Thus we have the following result.

Theorem 44 (Rank Theorem) When the map f has a finite-dimensional codomain, the rank of f^{\top} equals the rank of f.

Exercise 3.4 Let f be a map from one finite-dimensional vector space to another. Then, making the standard identifications, $(f^{\top})^{\top} = f$ and the kernel of f is the annihilator of the image of f^{\top} .

Exercise 3.5 For $\Phi \subset \mathcal{V}^{\top}$, let Φ^{\square} denote the subset of \mathcal{V} that contains all the vectors v such that $\varphi(v) = 0$ for all $\varphi \in \Phi$. Prove analogs of Theorems 42 and 43. Finally prove that when the map f has a finite-dimensional domain, the rank of f^{\top} equals the rank of f (another Rank Theorem).

3.5 The Contragredient

Suppose that \mathcal{V} is a finite-dimensional vector space with basis \mathcal{B} and dual basis \mathcal{B}^{\top} , and suppose that \mathcal{W} is an alias of \mathcal{V} via the isomorphism $f: \mathcal{V} \to \mathcal{W}$ which then sends the basis \mathcal{B} to the basis $f(\mathcal{B})$. Then the **contragredient** $f^{-\top} = (f^{-1})^{\top} = (f^{\top})^{-1}$ maps \mathcal{V}^{\top} isomorphically onto \mathcal{W}^{\top} and maps \mathcal{B}^{\top} onto the dual of the basis $f(\mathcal{B})$.

Theorem 45 Let $f: \mathcal{V} \to \mathcal{W}$ be an isomorphism between finite-dimensional vector spaces. Let \mathcal{B} be a basis for \mathcal{V} . Then the dual of the basis $f(\mathcal{B})$ is $f^{-\top}(\mathcal{B}^{\top})$.

Proof: Let
$$x, y \in \mathcal{B}$$
. We then have $\left(f^{-\top}\left(x^{\top}\right)\right)\left(f\left(y\right)\right) = \left(x^{\top} \circ f^{-1}\right)\left(f\left(y\right)\right) = x^{\top}\left(y\right)$.

This result is most often applied in the case when $\mathcal{V} = \mathcal{W}$ and the isomorphism f (now an **automorphism**) is specifically being used to effect a change of basis and the contragredient is being used to effect the corresponding change of the dual basis.

3.6 Product Spaces

The function space concept introduced at the beginning of this chapter can be profitably generalized. Instead of considering functions each value of which lies in the same vector space \mathcal{W} , we are now going to allow different vector spaces (all over the same field) for different elements of the domain \mathcal{D} . Thus for each $x \in \mathcal{D}$ there is to be a vector space \mathcal{W}_x , with each of these \mathcal{W}_x being over the same field, and we form all functions f such that $f(x) \in \mathcal{W}_x$. The set of all such f is the familiar Cartesian product of the \mathcal{W}_x . Any such Cartesian product of vector spaces may be made into a vector space using pointwise operations in the same way that we earlier used them to make $\mathcal{W}^{\mathcal{D}}$ into a function space, and it is this vector space that we mean whenever we refer to the Cartesian product of vector spaces as a **product space**. Sometimes we also refer to a product space or a Cartesian product simply as a **product**,

when there is no chance of confusion. We use the general notation $\prod_{x \in \mathcal{D}} \mathcal{W}_x$ for a product, but for the product of n vector spaces $\mathcal{W}_1, \ldots, \mathcal{W}_n$ we often write $\mathcal{W}_1 \times \cdots \times \mathcal{W}_n$. For a field \mathcal{F} , we recognize $\mathcal{F} \times \cdots \times \mathcal{F}$ (with n factors) as the familiar \mathcal{F}^n .

When there are only a finite number of factors, a product space is the direct sum of the "internalized" factors. For example, $W_1 \times W_2 = (W_1 \times \{0\}) \oplus (\{0\} \times W_2)$.

Exercise 3.6 If \mathcal{U}_1 and \mathcal{U}_2 are subspaces of the same vector space and have only 0 in common, then $\mathcal{U}_1 \times \mathcal{U}_2 \cong \mathcal{U}_1 \oplus \mathcal{U}_2$ in a manner not requiring any choice of basis.

However, when there are an infinite number of factors, a product space is not equal to the direct sum of "internalized" factors.

Exercise 3.7 Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of natural numbers and let \mathcal{F} be a field. Then the function space $\mathcal{F}^{\mathbb{N}}$, which is the same thing as the product space with each factor equal to \mathcal{F} and with one such factor for each natural number, is not equal to its subspace $\bigoplus_{i \in \mathbb{N}} \langle x_i \rangle$ where $x_i(j)$ is equal to 1 if i = j and is equal to 0 otherwise.

Selecting from a product space only those functions that have finitely many nonzero values gives a subspace called the **weak product space**. As an example, the set of all formal power series with coefficients in a field \mathcal{F} is a product space (isomorphic to the $\mathcal{F}^{\mathbb{N}}$ of the exercise above) for which the corresponding weak product space is the set of all polynomials with coefficients in \mathcal{F} . The weak product space always equals the direct sum of the "internalized" factors, and for that reason is also called the **external direct sum**. We will write $\biguplus_{x \in \mathcal{D}} \mathcal{W}_x$ for the weak product space derived from the product $\prod_{x \in \mathcal{D}} \mathcal{W}_x$.

3.7 Maps Involving Products

We single out some basic families of maps that involve products and their factors. The function π_x that sends $f \in \prod_{x \in \mathcal{D}} \mathcal{W}_x$ (or $\biguplus_{x \in \mathcal{D}} \mathcal{W}_x$) to $f(x) \in \mathcal{W}_x$, is called the **canonical projection** onto \mathcal{W}_x . The function η_x that sends $v \in \mathcal{W}_x$ to the $f \in \prod_{x \in \mathcal{D}} \mathcal{W}_x$ (or $\biguplus_{x \in \mathcal{D}} \mathcal{W}_x$) such that f(x) = v and f(y) = 0 for $y \neq x$, is called the **canonical injection** from \mathcal{W}_x . It is easy to verify that π_x and η_x are maps. Also, we introduce $\xi_x = \eta_x \circ \pi_x$, called the

component projection for component x. Note that $\pi_x \circ \eta_x = id_x$ (identity on \mathcal{W}_x), $\pi_x \circ \eta_y = 0$ whenever $x \neq y$, and $\xi_x \circ \xi_x = \xi_x$. Note also that $\sum_{x \in \mathcal{D}} \xi_x(f) = f$ so that $\sum_{x \in \mathcal{D}} \xi_x$ is the identity function, where we have made the convention that the pointwise sum of any number of 0s is 0.

A map $F: \mathcal{V} \to \prod_{x \in \mathcal{D}} \mathcal{W}_x$ from a vector space \mathcal{V} into the product of the vector spaces \mathcal{W}_x is uniquely determined by the map projections $\pi_x \circ F$.

Theorem 46 There exists exactly one map $F: \mathcal{V} \to \prod_{x \in \mathcal{D}} \mathcal{W}_x$ such that $\pi_x \circ F = \varphi_x$ where for each $x \in \mathcal{D}$, $\varphi_x : \mathcal{V} \to \mathcal{W}_x$ is a prescribed map.

Proof: The function F that sends $v \in \mathcal{V}$ to the $f \in \prod_{x \in \mathcal{D}} \mathcal{W}_x$ such that $f(x) = \varphi_x(v)$ is readily seen to be a map such that $\pi_x \circ F = \varphi_x$. Suppose that the map $G: \mathcal{V} \to \prod_{x \in \mathcal{D}} \mathcal{W}_x$ satisfies $\pi_x \circ G = \varphi_x$ for each $x \in \mathcal{D}$. For $v \in \mathcal{V}$ let G(v) = g. Then $g(x) = \pi_x(g) = \pi_x(G(v)) = (\pi_x \circ G)(v) = \varphi_x(v) = f(x)$ and hence g = f. But then G(v) = f = F(v) so that G(v) = F(v) which shows the uniqueness of F.

For weak products, it is the maps from them, rather than the maps into them, that have the special property.

Theorem 47 There exists exactly one map $F: \biguplus_{x \in \mathcal{D}} \mathcal{W}_x \to \mathcal{V}$ such that $F \circ \eta_x = \varphi_x$ where for each $x \in \mathcal{D}$, $\varphi_x : \mathcal{W}_x \to \mathcal{V}$ is a prescribed map.

Proof: The function F that sends $f \in \biguplus_{x \in \mathcal{D}} \mathcal{W}_x$ to $\sum_{\{x \in \mathcal{D} | f(x) \neq 0\}} \varphi_x (f(x))$ is readily seen to be a map such that $F \circ \eta_x = \varphi_x$. Suppose that the map $G : \biguplus_{x \in \mathcal{D}} \mathcal{W}_x \to \mathcal{V}$ satisfies $G \circ \eta_x = \varphi_x$ for each $x \in \mathcal{D}$. Let $f \in \biguplus_{x \in \mathcal{D}} \mathcal{W}_x$. Then $f = \sum_{\{x \in \mathcal{D} | f(x) \neq 0\}} \eta_x (f(x))$ so that $G(f) = \sum_{\{x \in \mathcal{D} | f(x) \neq 0\}} G(\eta_x (f(x))) = \sum_{\{x \in \mathcal{D} | f(x) \neq 0\}} \varphi_x (f(x)) = F(f)$, which shows the uniqueness of F.

Exercise 3.8 $\biguplus_{x \in \mathcal{D}} \mathcal{W}_x = \bigoplus_{x \in \mathcal{D}} \eta_x (\mathcal{W}_x)$.

Exercise 3.9 $\bigoplus_{x \in \mathcal{D}} \mathcal{U}_x \cong \biguplus_{x \in \mathcal{D}} \mathcal{U}_x$ in a manner not requiring any choice of basis.

Exercise 3.10 Let W_1, \ldots, W_n be finite dimensional vector spaces over the same field. Then

$$\dim (\mathcal{W}_1 \times \cdots \times \mathcal{W}_n) = \dim \mathcal{W}_1 + \cdots + \dim \mathcal{W}_n.$$

The last two theorems lay a foundation for two important isomorphisms. As before, with all vector spaces over the same field, let the vector space \mathcal{V} be given, and let a vector space \mathcal{W}_x be given for each x in some set \mathcal{D} . Then, utilizing map spaces, we may form the pairs of vector spaces

$$\mathcal{M} = \prod_{x \in \mathcal{D}} \left\{ \mathcal{V}
ightarrow \mathcal{W}_x
ight\}, \, \mathcal{N} = \left\{ \mathcal{V}
ightarrow \prod_{x \in \mathcal{D}} \mathcal{W}_x
ight\}$$

and

$$\mathcal{M}^{'} = \prod_{x \in \mathcal{D}} \left\{ \mathcal{W}_x
ightarrow \mathcal{V}
ight\}, \, \mathcal{N}^{'} = \left\{ igoplus_{x \in \mathcal{D}} \mathcal{W}_x
ightarrow \mathcal{V}
ight\}.$$

Theorem 48 $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{M}' \cong \mathcal{N}'$.

Proof:

 $f \in \mathcal{M}$ means that for each $x \in \mathcal{D}$, $f(x) = \varphi_x$ for some map $\varphi_x : \mathcal{V} \to \mathcal{W}_x$, and hence there exists exactly one map $F \in \mathcal{N}$ such that $\pi_x \circ F = f(x)$ for each $x \in \mathcal{D}$. This association of F with f therefore constitutes a well-defined function $\Lambda : \mathcal{M} \to \mathcal{N}$ which is easily seen to be a map. The map Λ is one-to-one. For if Λ sends both f and g to F, then for each $x \in \mathcal{D}$, $\pi_x \circ F = f(x) = g(x)$, and f and g must therefore be equal. Also, each $F \in \mathcal{N}$ is the image under Λ of the $f \in \mathcal{M}$ such that $f(x) = \pi_x \circ F$ so that Λ is also onto. Hence Λ is an isomorphism from \mathcal{M} onto \mathcal{N} .

A similar argument shows that $\mathcal{M}' \cong \mathcal{N}'$.

Exercise 3.11 $(\biguplus_{x \in \mathcal{D}} \mathcal{W}_x)^{\top} \cong \prod_{x \in \mathcal{D}} \mathcal{W}_x^{\top}$. (Compare with Example 40 at the beginning of this chapter.)

Exercise 3.12
$$(W_1 \times \cdots \times W_n)^{\top} \cong W_1^{\top} \times \cdots \times W_n^{\top}$$
.

We note that the isomorphisms of the two exercises and theorem above do not depend on any choice of basis. This will generally be the case for the isomorphisms we will be establishing. From now on, we will usually skip pointing out when an isomorphism does not depend on specific choices, but will endeavor to point out any opposite cases.

Up to isomorphism, the order of the factors in a product does not matter. This can be proved quite readily directly from the product definition, but the theorem above on maps into a product gives an interesting approach from another perspective. We also incorporate the "obvious" result that using aliases of the factors does not affect the product, up to isomorphism.

Theorem 49 $\prod_{x \in \mathcal{D}} \mathcal{W}_x \cong \prod_{x \in \mathcal{D}} \mathcal{W}'_{\sigma(x)}$ where σ is a bijection of \mathcal{D} onto itself, and for each $x \in \mathcal{D}$ there is an isomorphism $\theta_x : \mathcal{W}'_x \to \mathcal{W}_x$.

Proof: In Theorem 46, take the product to be $\prod_{x\in\mathcal{D}}\mathcal{W}_x$, take $\mathcal{V}=\prod_{x\in\mathcal{D}}\mathcal{W}'_{\sigma(x)}$, and take $\varphi_x=\theta_x\circ\pi_{\sigma^{-1}(x)}$ so that there exists a map Ψ from $\prod_{x\in\mathcal{D}}\mathcal{W}'_{\sigma(x)}$ to $\prod_{x\in\mathcal{D}}\mathcal{W}_x$ such that $\pi_x\circ\Psi=\theta_x\circ\pi_{\sigma^{-1}(x)}$. Interchanging the product spaces and applying the theorem again, there exists a map Φ from $\prod_{x\in\mathcal{D}}\mathcal{W}_x$ to $\prod_{x\in\mathcal{D}}\mathcal{W}'_{\sigma(x)}$ such that $\pi_{\sigma^{-1}(x)}\circ\Phi=\theta_x^{-1}\circ\pi_x$. Then $\pi_x\circ\Psi\circ\Phi=\theta_x\circ\pi_{\sigma^{-1}(x)}\circ\Phi=\pi_x$, and $\pi_{\sigma^{-1}(x)}\circ\Phi\circ\Psi=\theta_x^{-1}\circ\pi_x\circ\Psi=\pi_{\sigma^{-1}(x)}$. Now applying the theorem to the two cases where the \mathcal{V} and the product space are the same, the unique map determined in each case must be the identity. Hence $\Psi\circ\Phi$ and $\Phi\circ\Psi$ must each be identity maps, thus making Φ and Ψ a pair of mutually inverse maps that link $\prod_{x\in\mathcal{D}}\mathcal{W}_x$ and $\prod_{x\in\mathcal{D}}\mathcal{W}'_{\sigma(x)}$ isomorphically. \blacksquare

3.8 Problems

- 1. Let \mathcal{V} be a vector space over the field \mathcal{F} . Let $f \in \mathcal{V}^{\top}$. Then f^{\top} is the map that sends the element φ of $(\mathcal{F}^1)^{\top}$ to $(\varphi(1)) \cdot f \in \mathcal{V}^{\top}$.
- 2. Let $\mathcal{V} \lhd \mathcal{W}$. Let $j : \mathcal{V} \to \mathcal{W}$ be the inclusion map of \mathcal{V} into \mathcal{W} which sends elements in \mathcal{V} to themselves. Then j^{\top} is the corresponding "restriction of domain" operation on the functionals of \mathcal{W}^{\top} , restricting each of their domains to \mathcal{V} from the original \mathcal{W} . (Thus, using a frequently-seen notation, $j^{\top}(\varphi) = \varphi_{|\mathcal{V}}$ for any $\varphi \in \mathcal{W}^{\top}$.)
 - 3. Let $f: \mathcal{V} \to \mathcal{W}$ be a map. Then for all $w \in \mathcal{W}$ the equation

$$f(x) = w$$

has a unique solution for x if and only if the equation

$$f(x) = 0$$

has only the trivial solution x = 0. (This result is sometimes called the *Alternative Theorem*. However, see the next problem.)

4. (Alternative Theorem) Let \mathcal{V} and \mathcal{W} be finite-dimensional and let $f: \mathcal{V} \to \mathcal{W}$ be a map. Then for a given w there exists x such that

$$f\left(x\right) = w$$

if and only if

$$w \in \left(\operatorname{Kernel} \left(f^{\top} \right) \right)^{0}$$
,

- i. e., if and only if $\varphi(w) = 0$ for all $\varphi \in \text{Kernel}(f^{\top})$. (The reason this is called the *Alternative* Theorem is because it is usually phrased in the equivalent form: *Either* the equation f(x) = w can be solved for x, or there is a $\varphi \in \text{Kernel}(f^{\top})$ such that $\varphi(w) \neq 0$.)
- 5. Let \mathcal{V} and \mathcal{W} be finite-dimensional and let $f: \mathcal{V} \to \mathcal{W}$ be a map. Then f is onto if and only if f^{\top} is one-to-one and f^{\top} is onto if and only if f is one-to-one.
- 6. Let \mathcal{V} and \mathcal{W} have the respective finite dimensions m and n over the finite field \mathcal{F} of q elements. How many maps $\mathcal{V} \to \mathcal{W}$ are there? If $m \ge n$ how many of these are onto, if $m \le n$ how many are one-to-one, and if m = n how many are isomorphisms?

4 Multilinearity and Tensoring

4.1 Multilinear Transformations

Maps, or linear transformations, are functions of a single vector variable. Also important are the multilinear transformations. These are the functions of several vector variables (i. e., functions on the product space $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n$) which are linear in each variable separately. In detail, we call the function $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$, where the \mathcal{V}_i and \mathcal{W} are vector spaces over the same field \mathcal{F} , a **multilinear transformation**, if for each k

$$f(v_1, \dots, v_{k-1}, a \cdot u + b \cdot v, \dots, v_n) =$$

$$= a \cdot f(v_1, \dots, v_{k-1}, u, \dots, v_n) + b \cdot f(v_1, \dots, v_{k-1}, v, \dots, v_n)$$

for all vectors u, v, v_1, \ldots, v_n and all scalars a, b. We also say that such an f is n-linear (bilinear, trilinear when n = 2, 3). When $\mathcal{W} = \mathcal{F}$, we call a multilinear transformation a multilinear functional. In general, a multilinear transformation is not a map on $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n$. However, we will soon construct a vector space on which the linear transformations "are" the multilinear transformations on $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n$.

A unique multilinear transformation is determined by the arbitrary assignment of values on all tuples of basis vectors.

Theorem 50 Let V_1, \ldots, V_n, W be vector spaces over the same field and for each V_i let the basis \mathcal{B}_i be given. Then given the function $f_0: \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \to \mathcal{W}$, there is a unique multilinear transformation $f: V_1 \times \cdots \times V_n \to \mathcal{W}$ such that f agrees with f_0 on $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$.

Proof: Given $v_i \in \mathcal{V}_i$ there is a unique finite subset \mathcal{X}_i of \mathcal{B}_i and unique nonzero scalars a_{x_i} such that $v_i = \sum_{x_i \in \mathcal{X}_i} a_{x_i} \cdot x_i$. The *n*-linearity of $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ implies

$$f(v_1,\ldots,v_n) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_n \in \mathcal{X}_n} a_{x_1} \cdots a_{x_n} \cdot f(x_1,\ldots,x_n).$$

Hence for f to agree with f_0 on $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, it can only have the value

$$f\left(v_{1},\ldots,v_{n}\right)=\sum_{x_{1}\in\mathcal{X}_{1}}\cdots\sum_{x_{n}\in\mathcal{X}_{n}}a_{x_{1}}\cdots a_{x_{n}}\cdot f_{0}\left(x_{1},\ldots,x_{n}\right).$$

On the other hand, setting

$$f(v_1, \dots, v_n) = \sum_{x_1 \in \mathcal{X}_1} \dots \sum_{x_n \in \mathcal{X}_n} a_{x_1} \dots a_{x_n} \cdot f_0(x_1, \dots, x_n)$$

does define a function $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ which clearly agrees with f_0 on $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$, and this f is readily verified to be n-linear.

4.2 Functional Product Spaces

Let S_1, \ldots, S_n be sets of linear functionals on the respective vector spaces V_1, \ldots, V_n all over the same field \mathcal{F} . By the **functional product** $S_1 \cdots S_n$ is meant the set of all products $f_1 \cdots f_n$ with each $f_i \in S_i$, where by such a product of linear functionals is meant the function $f_1 \cdots f_n : V_1 \times \cdots \times V_n \to \mathcal{F}$ such that $f_1 \cdots f_n (v_1, \ldots, v_n) = f_1(v_1) \cdots f_n(v_n)$. The function-space subspace $\langle S_1 \cdots S_n \rangle$ obtained by taking all linear combinations of finite subsets of the functional product $S_1 \cdots S_n$ is called the **functional product space** generated by $S_1 \cdots S_n$.

Exercise 4.1 Each element of $\langle \mathcal{S}_1 \cdots \mathcal{S}_n \rangle$ is a multilinear functional on $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n$.

Exercise 4.2
$$\langle\langle \mathcal{S}_1 \rangle \cdots \langle \mathcal{S}_n \rangle\rangle = \langle \mathcal{S}_1 \cdots \mathcal{S}_n \rangle$$
.

Lemma 51 Let W_1, \ldots, W_n be vector spaces of linear functionals, all over the same field, with respective bases $\mathcal{B}_1, \ldots, \mathcal{B}_n$. Then $\mathcal{B}_1 \cdots \mathcal{B}_n$ is an independent set.

Proof: Induction will be employed. The result is clearly true for n=1, and suppose that it is true for n=N-1. Let \mathcal{X} be a finite subset of $\mathcal{B}_1\cdots\mathcal{B}_N$ such that $\sum_{x\in\mathcal{X}}a_x\cdot x=0$. Each $x\in\mathcal{X}$ has the form yz where $y\in\mathcal{B}_1\cdots\mathcal{B}_{N-1}$ and $z\in\mathcal{B}_N$. Collecting terms on the distinct z, $\sum_{x\in\mathcal{X}}a_x\cdot x=\sum_z(\sum_{x=yz}a_x\cdot y)z$. The independence of \mathcal{B}_N implies that for each $z,\sum_{x=yz}a_x\cdot y=0$, and the assumed independence of $\mathcal{B}_1\cdots\mathcal{B}_{N-1}$ then implies that each a_x is zero. Hence $\mathcal{B}_1\cdots\mathcal{B}_N$ is an independent set, and by induction, $\mathcal{B}_1\cdots\mathcal{B}_n$ is an independent set for each n.

Proposition 52 Let $W_1, ..., W_n$ be vector spaces of linear functionals, all over the same field, with respective bases $\mathcal{B}_1, ..., \mathcal{B}_n$. Then $\mathcal{B}_1 \cdots \mathcal{B}_n$ is a basis for $\langle W_1 \cdots W_n \rangle$.

Proof: By the exercise above, $\langle W_1 \cdots W_n \rangle = \langle \mathcal{B}_1 \cdots \mathcal{B}_n \rangle$ and hence $\mathcal{B}_1 \cdots \mathcal{B}_n$ spans $\langle W_1 \cdots W_n \rangle$. By the lemma above, $\mathcal{B}_1 \cdots \mathcal{B}_n$ is an independent set. Thus $\mathcal{B}_1 \cdots \mathcal{B}_n$ is a basis for $\langle W_1 \cdots W_n \rangle$.

4.3 Functional Tensor Products

In the previous chapter, it was shown that the vector space \mathcal{V} may be embedded in its double dual $\mathcal{V}^{\top\top}$ independently of any choice of basis, effectively making each vector in \mathcal{V} into a linear functional on \mathcal{V}^{\top} . Thus given the vector spaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$, all over the same field \mathcal{F} , and the natural injection Θ_i that embeds each \mathcal{V}_i in its double dual, we may form the functional product space $\langle \Theta_1(\mathcal{V}_1) \cdots \Theta_n(\mathcal{V}_n) \rangle$ which will be called the **functional tensor product** of the \mathcal{V}_i and which will be denoted by $\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n$. Similarly, an element $\Theta_1(v_1) \cdots \Theta_n(v_n)$ of $\Theta_1(\mathcal{V}_1) \cdots \Theta_n(\mathcal{V}_n)$ will be called the **functional tensor product** of the v_i and will be denoted by $v_1 \bigotimes \cdots \bigotimes v_n$.

If W is a vector space over the same field as the V_i , the "universal" multilinear transformation $\Xi: V_1 \times \cdots \times V_n \to V_1 \bigotimes \cdots \bigotimes V_n$ which sends (v_1, \ldots, v_n) to $v_1 \otimes \cdots \otimes v_n$ exchanges the multilinear transformations $f: V_1 \times \cdots \times V_n \to W$ with the linear transformations $\varphi: V_1 \bigotimes \cdots \bigotimes V_n \to W$ via the composition of functions $f = \varphi \circ \Xi$.

Theorem 53 Let V_1, \ldots, V_n, W be vector spaces over the same field. Then for each multilinear transformation $f: V_1 \times \cdots \times V_n \to W$ there exists a unique linear transformation $\varphi: V_1 \otimes \cdots \otimes V_n \to W$ such that $f = \varphi \circ \Xi$, where $\Xi: V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$ is the tensor product function that sends (v_1, \ldots, v_n) to $v_1 \otimes \cdots \otimes v_n$.

Proof: For each \mathcal{V}_i , choose some basis \mathcal{B}_i . Then by the proposition above, $\mathcal{B} = \{x_1 \otimes \cdots \otimes x_n \mid x_1 \in \mathcal{B}_1, \ldots, x_n \in \mathcal{B}_n\}$ is a basis for $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$. Given the multilinear transformation $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$, consider the linear transformation $\varphi: \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n \to \mathcal{W}$ such that for $x_1 \in \mathcal{B}_1, \ldots, x_n \in \mathcal{B}_n$

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi(\Xi(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n).$$

Since $\varphi \circ \Xi$ is clearly *n*-linear, it must then equal f by the theorem above. Every φ determines an *n*-linear f via $f = \varphi \circ \Xi$, but since the values of φ on \mathcal{B} determine it completely, and the values of f on $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ determine it completely, a different φ determines a different f. Thus, while multilinear transformations themselves are generally not maps on $\mathcal{V}_1 \times \cdots \times \mathcal{V}_n$, they do correspond one-to-one to the maps on the related vector space $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n = \langle \{v_1 \otimes \cdots \otimes v_n\} \rangle$. In fact, we can interpret this correspondence as an isomorphism if we introduce the function space subspace $\left\{ \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \stackrel{(n-linear)}{\to} \mathcal{W} \right\}$ of *n*-linear functions, which is then clearly isomorphic to the map space $\left\{ \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n \to \mathcal{W} \right\}$. The vector space $\left\{ \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \stackrel{(n-linear)}{\to} \mathcal{F} \right\}$ of multilinear functionals is therefore isomorphic to $(\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n)^{\top}$. If all the \mathcal{V}_i are finite-dimensional, we then have $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n \cong \left\{ \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \stackrel{(n-linear)}{\to} \mathcal{F} \right\}^{\top}$.

When the \mathcal{V}_i are finite-dimensional, we may discard any even number of consecutive dualization operators, since we identify a finite-dimensional space with its double dual. Therefore, $\mathcal{V}_1^{\top} \otimes \cdots \otimes \mathcal{V}_n^{\top}$ is just $\langle \mathcal{V}_1^{\top} \cdots \mathcal{V}_n^{\top} \rangle$. Thus $\mathcal{V}_1^{\top} \otimes \cdots \otimes \mathcal{V}_n^{\top}$ is a subspace of $\{\mathcal{V}_1 \times \cdots \times \mathcal{V}_n \overset{(n-linear)}{\to} \mathcal{F}\}$, and it is easy to see that both have the same dimension, so they are actually equal. This and our observations immediately above then give us the following result.

Theorem 54 Let V_1, \ldots, V_n be vector spaces over the field \mathcal{F} . Then

$$(\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n)^{\top} \cong \left\{ \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \stackrel{(n-linear)}{\rightarrow} \mathcal{F} \right\}.$$

If the V_i are all finite-dimensional, we also have

$$\mathcal{V}_1 igotimes \cdots igotimes \mathcal{V}_n \cong \left\{ \mathcal{V}_1 imes \cdots imes \mathcal{V}_n \stackrel{(n-linear)}{
ightarrow} \mathcal{F}
ight\}^ op$$

and

$$(\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n)^\top \cong \mathcal{V}_1^\top \bigotimes \cdots \bigotimes \mathcal{V}_n^\top = \left\{ \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \stackrel{(n-linear)}{\to} \mathcal{F} \right\} : EndProof$$

4.4 Tensor Products in General

Theorem 53 contains the essence of the tensor product concept. By making a definition out of the *universal property* that this theorem ascribes to $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ and Ξ , a general tensor product concept results. Let the vector spaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ each over the same field \mathcal{F} be given. A **tensor product** of $\mathcal{V}_1, \ldots, \mathcal{V}_n$ (in that order) is a vector space Π (a **tensor product space**)

over \mathcal{F} along with an n-linear function $\Upsilon: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \Pi$ (a **tensor product function**) such that given any vector space \mathcal{W} , every n-linear function $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ is equal to $\varphi \circ \Upsilon$ for a unique map $\varphi: \Pi \to \mathcal{W}$. The functional tensor product $\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n$ along with the functional tensor product function Ξ are thus an example of a tensor product of $\mathcal{V}_1, \ldots, \mathcal{V}_n$.

Another tensor product of the same factor spaces may be defined using any alias of a tensor product space along with the appropriate tensor product function.

Theorem 55 Let V_1, \ldots, V_n be vector spaces over the same field. Then if Π along with the n-linear function $\Upsilon : V_1 \times \cdots \times V_n \to \Pi$ is a tensor product of V_1, \ldots, V_n , and $\Theta : \Pi \to \Pi'$ is an isomorphism, then Π' along with $\Theta \circ \Upsilon$ is also a tensor product of V_1, \ldots, V_n .

Proof: Given the vector space \mathcal{W} , the *n*-linear function $f: \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ is equal to $\varphi \circ \Upsilon$ for a unique map $\varphi: \Pi \to \mathcal{W}$. The map $\varphi \circ \Theta^{-1}$ therefore satisfies $(\varphi \circ \Theta^{-1}) \circ (\Theta \circ \Upsilon) = f$. On the other hand, if $\varphi' \circ (\Theta \circ \Upsilon) = (\varphi' \circ \Theta) \circ \Upsilon = f$, then by the uniqueness of φ , $\varphi' \circ \Theta = \varphi$ and hence $\varphi' = \varphi \circ \Theta^{-1}$.

Exercise 4.3 Make sense of the tensor product concept in the case where n = 1, so that we have 1-linearity and the tensor product of a single factor.

If in the manner of the previous theorem, we have isomorphic tensor product spaces with the tensor product functions related by the isomorphism as in the theorem, we say that the tensor products are **linked** by the isomorphism of their tensor product spaces. We now show that tensor products of the same factors always have isomorphic tensor product spaces, and in fact there is a unique isomorphism between them that links them.

Theorem 56 Let V_1, \ldots, V_n be vector spaces over the same field. Then if Π along with the n-linear function $\Upsilon: V_1 \times \cdots \times V_n \to \Pi$, and Π' along with the n-linear function $\Upsilon': V_1 \times \cdots \times V_n \to \Pi'$, are each tensor products of V_1, \ldots, V_n , then there is a unique isomorphism $\Theta: \Pi \to \Pi'$ such that $\Theta \circ \Upsilon = \Upsilon'$.

Proof: In the tensor product definition above, take $W = \Pi'$ and $f = \Upsilon'$. Then there is a unique map $\Theta : \Pi \to \Pi'$ such that $\Upsilon' = \Theta \circ \Upsilon$. Similarly, there is a unique map $\Theta' : \Pi' \to \Pi$ such that $\Upsilon = \Theta' \circ \Upsilon'$. Hence $\Upsilon' = \Theta \circ \Theta' \circ \Upsilon'$ and $\Upsilon = \Theta' \circ \Theta \circ \Upsilon$. Applying the tensor product definition again, the unique map $\varphi : \Pi \to \Pi$ such that $\Upsilon = \varphi \circ \Upsilon$ must be the identity. We conclude that $\Theta' \circ \Theta$ is the identity and the same for $\Theta \circ \Theta'$. Hence the unique map Θ has the inverse Θ' and is therefore an isomorphism.

We will write any tensor product space as $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ just as we did for the functional tensor product. Similarly we will write $v_1 \otimes \cdots \otimes v_n$ for $\Upsilon(v_1, \ldots, v_n)$, and call it the **tensor product** of the vectors v_1, \ldots, v_n , just as we did for the functional tensor product $\Xi(v_1, \ldots, v_n)$. With isomorphic tensor product spaces, we will always assume that the tensor products are linked by the isomorphism. By assuming this, tensor products of the same vectors will then correspond among all tensor products of the same factor spaces: $v_1 \otimes \cdots \otimes v_n$ in one of these spaces is always an isomorph of $v_1 \otimes \cdots \otimes v_n$ in any of the others.

Exercise 4.4 A tensor product space is spanned by the image of its associated tensor product function: $\langle \{v_1 \otimes \cdots \otimes v_n \mid v_1 \in \mathcal{V}_1, \dots, v_n \in \mathcal{V}_n\} \rangle = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$.

Up to isomorphism, the order of the factors in a tensor product does not matter. We also incorporate the "obvious" result that using aliases of the factors does not affect the result, up to isomorphism.

Theorem 57 Let the vector spaces $\mathcal{V}'_1, \ldots, \mathcal{V}'_n$, all over the same field, be (possibly reordered) aliases of $\mathcal{V}_1, \ldots, \mathcal{V}_n$, and let $\mathcal{V}_{\otimes} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$ along with the tensor product function $\Upsilon: \mathcal{V}_{\times} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{V}_{\otimes}$ be a tensor product, and let $\mathcal{V}'_{\otimes} = \mathcal{V}'_1 \otimes \cdots \otimes \mathcal{V}'_n$ along with the tensor product function $\Upsilon: \mathcal{V}'_{\times} = \mathcal{V}'_1 \times \cdots \times \mathcal{V}'_n \to \mathcal{V}'_{\otimes}$ be a tensor product. Then $\mathcal{V}_{\otimes} \cong \mathcal{V}'_{\otimes}$.

Proof: Let $\Phi: \mathcal{V}_{\times} \to \mathcal{V}'_{\times}$ denote the isomorphism that exists from the product space \mathcal{V}_{\times} to the product space \mathcal{V}'_{\times} . It is easy to see that $\Upsilon' \circ \Phi$ and $\Upsilon \circ \Phi^{-1}$ are n-linear. Hence there is a (unique) map $\Theta: \mathcal{V}_{\otimes} \to \mathcal{V}'_{\otimes}$ such that $\Theta \circ \Upsilon = \Upsilon' \circ \Phi$ and there is a (unique) map $\Theta': \mathcal{V}'_{\otimes} \to \mathcal{V}_{\otimes}$ such that $\Theta' \circ \Upsilon' = \Upsilon \circ \Phi^{-1}$. From this we readily deduce that $\Upsilon' = \Theta \circ \Theta' \circ \Upsilon'$ and $\Upsilon = \Theta' \circ \Theta \circ \Upsilon$. Hence the map Θ has the inverse Θ' and is therefore an isomorphism. \blacksquare

Up to isomorphism, tensor multiplication of vector spaces may be performed iteratively.

Theorem 58 Let V_1, \ldots, V_n be vector spaces over the same field. Then for any integer k such that $1 \leq k < n$ there is an isomorphism

$$\Theta: (\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_k) \bigotimes (\mathcal{V}_{k+1} \bigotimes \cdots \bigotimes \mathcal{V}_n) \to \mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n$$

such that

$$\Theta\left((v_1\otimes\cdots\otimes v_k)\otimes(v_{k+1}\otimes\cdots\otimes v_n)\right)=v_1\otimes\cdots\otimes v_n.$$

Proof: Set $\mathcal{V}_{\times} = \mathcal{V}_{1} \times \cdots \times \mathcal{V}_{n}$, $\mathcal{V}_{\otimes} = \mathcal{V}_{1} \otimes \cdots \otimes \mathcal{V}_{n}$, $\overline{\mathcal{V}}_{\times} = \mathcal{V}_{1} \times \cdots \times \mathcal{V}_{k}$, $\overline{\mathcal{V}}_{\otimes} = \mathcal{V}_{1} \otimes \cdots \otimes \mathcal{V}_{k}$, $\overline{\overline{\mathcal{V}}}_{\times} = \mathcal{V}_{k+1} \times \cdots \times \mathcal{V}_{n}$, and $\overline{\overline{\mathcal{V}}}_{\otimes} = \mathcal{V}_{k+1} \otimes \cdots \otimes \mathcal{V}_{n}$.

For fixed $v_{k+1} \otimes \cdots \otimes v_n \in \overline{\overline{\mathcal{V}}}_{\otimes}$ we define the k-linear function $\overline{f}_{v_{k+1} \otimes \cdots \otimes v_n}$: $\overline{\mathcal{V}}_{\times} \to \mathcal{V}_{\otimes}$ by $\overline{f}_{v_{k+1} \otimes \cdots \otimes v_n} (v_1, \ldots, v_k) = v_1 \otimes \cdots \otimes v_n$. Corresponding to $\overline{f}_{v_{k+1} \otimes \cdots \otimes v_n}$ is the (unique) map $\overline{\Theta}_{v_{k+1} \otimes \cdots \otimes v_n} : \overline{\mathcal{V}}_{\otimes} \to \mathcal{V}_{\otimes}$ such that

$$\overline{\Theta}_{v_{k+1}\otimes\cdots\otimes v_n}\left(v_1\otimes\cdots\otimes v_k\right)=v_1\otimes\cdots\otimes v_n.$$

Similarly, for fixed $v_1 \otimes \cdots \otimes v_k \in \overline{\mathcal{V}}_{\otimes}$ we define the (n-k)-linear function $\overline{f}_{v_1 \otimes \cdots \otimes v_k} : \overline{\overline{\mathcal{V}}}_{\times} \to \mathcal{V}_{\otimes}$ by $\overline{\overline{f}}_{v_1 \otimes \cdots \otimes v_k} (v_{k+1}, \dots, v_n) = v_1 \otimes \cdots \otimes v_n$. Corresponding to $\overline{\overline{f}}_{v_1 \otimes \cdots \otimes v_k}$ is the (unique) map $\overline{\overline{\Theta}}_{v_1 \otimes \cdots \otimes v_k} : \overline{\overline{\mathcal{V}}}_{\otimes} \to \mathcal{V}_{\otimes}$ such that

$$\overline{\overline{\Theta}}_{v_1 \otimes \cdots \otimes v_k} (v_{k+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n.$$

We then define the function $f: \overline{\mathcal{V}}_{\otimes} \times \overline{\overline{\mathcal{V}}}_{\otimes} \to \mathcal{V}_{\otimes}$ by the formula

$$f\left(x,y\right) = \sum a_{v_{1} \otimes \cdots \otimes v_{k}} \overline{\overline{\Theta}}_{v_{1} \otimes \cdots \otimes v_{k}}\left(y\right)$$

when $x = \sum a_{v_1 \otimes \cdots \otimes v_k} \cdot v_1 \otimes \cdots \otimes v_k$. We claim this formula gives the same result no matter how x is expressed as a linear combination of elements of the form $v_1 \otimes \cdots \otimes v_k$. To aid in verifying this, let y be expressed as $y = \sum b_{v_{k+1} \otimes \cdots \otimes v_n} \cdot v_{k+1} \otimes \cdots \otimes v_n$. The formula may then be written

$$f(x,y) = \sum a_{v_1 \otimes \cdots \otimes v_k} \cdot \sum b_{v_{k+1} \otimes \cdots \otimes v_n} \cdot \overline{\overline{\Theta}}_{v_1 \otimes \cdots \otimes v_k} (v_{k+1} \otimes \cdots \otimes v_n).$$

But $\overline{\overline{\Theta}}_{v_1 \otimes \cdots \otimes v_k} (v_{k+1} \otimes \cdots \otimes v_n) = \overline{\Theta}_{v_{k+1} \otimes \cdots \otimes v_n} (v_1 \otimes \cdots \otimes v_k)$ so that the formula is equivalent to

$$f\left(x,y\right) = \sum a_{v_{1}\otimes\cdots\otimes v_{k}}\cdot\sum b_{v_{k+1}\otimes\cdots\otimes v_{n}}\cdot\overline{\Theta}_{v_{k+1}\otimes\cdots\otimes v_{n}}\left(v_{1}\otimes\cdots\otimes v_{k}\right).$$

But this is the same as

$$f\left(x,y\right) = \sum b_{v_{k+1}\otimes\cdots\otimes v_{n}}\cdot\overline{\Theta}_{v_{k+1}\otimes\cdots\otimes v_{n}}\left(x\right)$$

which of course has no dependence on how x is expressed as a linear combination of elements of the form $v_1 \otimes \cdots \otimes v_k$.

f is clearly linear in either argument when the other argument is held fixed. Corresponding to f is the (unique) map $\Theta: \overline{\mathcal{V}}_{\otimes} \bigotimes \overline{\overline{\mathcal{V}}}_{\otimes} \to \mathcal{V}_{\otimes}$ such that $\Theta(x \otimes y) = f(x,y)$ so that, in particular,

$$\Theta\left((v_1\otimes\cdots\otimes v_k)\otimes(v_{k+1}\otimes\cdots\otimes v_n)\right)=v_1\otimes\cdots\otimes v_n.$$

We define an *n*-linear function $f': \mathcal{V}_{\times} \to \overline{\mathcal{V}}_{\otimes} \bigotimes \overline{\overline{\mathcal{V}}}_{\otimes}$ by setting

$$f'(v_1,\ldots,v_n)=(v_1\otimes\cdots\otimes v_k)\otimes(v_{k+1}\otimes\cdots\otimes v_n).$$

Corresponding to f' is the (unique) map $\Theta': \mathcal{V}_{\otimes} \to \overline{\mathcal{V}}_{\otimes} \bigotimes \overline{\overline{\mathcal{V}}}_{\otimes}$ such that

$$\Theta'(v_1 \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n).$$

Thus each of $\Theta' \circ \Theta$ and $\Theta \circ \Theta'$ coincides with the identity on a spanning set for its domain, so each is in fact the identity map. Hence the map Θ has the inverse Θ' and therefore is an isomorphism.

Exercise 4.5
$$(\cdots((\mathcal{V}_1 \otimes \mathcal{V}_2) \otimes \mathcal{V}_3) \otimes \cdots \otimes \mathcal{V}_{n-1}) \otimes \mathcal{V}_n \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$$
.

Corollary 59
$$(\mathcal{V}_1 \bigotimes \mathcal{V}_2) \bigotimes \mathcal{V}_3 \cong \mathcal{V}_1 \bigotimes (\mathcal{V}_2 \bigotimes \mathcal{V}_3) \cong \mathcal{V}_1 \bigotimes \mathcal{V}_2 \bigotimes \mathcal{V}_3$$
.

The two preceding theorems form the foundation for the following general associativity result.

Theorem 60 Tensor products involving the same vector spaces are isomorphic no matter how the factors are grouped.

Proof: Complete induction will be employed. The result is trivially true for one or two spaces involved. Suppose the result is true for r spaces for every r < n. Consider a grouped (perhaps nested) tensor product Π of the n > 2 vector spaces $\mathcal{V}_1, ..., \mathcal{V}_n$ in that order. At the outermost essential level of grouping, we have an expression of the form $\Pi = \mathcal{W}_1 \bigotimes \cdots \bigotimes \mathcal{W}_k$, where $1 < k \leqslant n$ and the n factor spaces $\mathcal{V}_1, ..., \mathcal{V}_n$ are distributed in (perhaps

nested) groupings among the W_i . Then $\Pi \cong (W_1 \otimes \cdots \otimes W_{k-1}) \otimes W_k$ and by the induction hypothesis we may assume that, for some i < n, $W_1 \otimes \cdots \otimes W_{k-1} \cong V_1 \otimes \cdots \otimes V_i$ and $W_k \cong V_{i+1} \otimes \cdots \otimes V_n$. Hence $\Pi \cong (V_1 \otimes \cdots \otimes V_i) \otimes (V_{i+1} \otimes \cdots \otimes V_n)$ and therefore $\Pi \cong V_1 \otimes \cdots \otimes V_n$ no matter how the n factors $V_1, ..., V_n$ are grouped. The theorem thus holds by induction. \blacksquare

Not surprisingly, when we use the field \mathcal{F} (considered as a vector space over itself) as a tensor product factor, it does not really do anything.

Theorem 61 Let V be a vector space over the field \mathcal{F} , and let \mathcal{F} be considered also to be a vector space over itself. Then there is an isomorphism from $\mathcal{F} \bigotimes \mathcal{V}$ (and another from $\mathcal{V} \bigotimes \mathcal{F}$) to \mathcal{V} sending $a \otimes v$ (and the other sending $v \otimes a$) to $a \cdot v$.

Proof: The function from $\mathcal{F} \times \mathcal{V}$ to \mathcal{V} that sends (a, v) to $a \cdot v$ is clearly bilinear. Hence there exists a (unique) map $\Theta : \mathcal{F} \bigotimes \mathcal{V} \to \mathcal{V}$ that sends $a \otimes v$ to $a \cdot v$. Let $\Theta' : \mathcal{V} \to \mathcal{F} \bigotimes \mathcal{V}$ be the map that sends v to $1 \otimes v$. Then $\Theta \circ \Theta'$ is clearly the identity on \mathcal{V} . On the other hand, $\Theta' \circ \Theta$ sends each element in $\mathcal{F} \bigotimes \mathcal{V}$ of the form $a \otimes v$ to itself, and so is the identity on $\mathcal{F} \bigotimes \mathcal{V}$.

The other part is proved similarly. \blacksquare

4.5 Problems

- 1. Suppose that in $\mathcal{V}_1 \bigotimes \mathcal{V}_2$ we have $u \otimes v = u \otimes w$ for some nonzero $u \in \mathcal{V}_1$. Is it then possible for the vectors v and w in \mathcal{V}_2 to be different?
- 2. When all \mathcal{V}_i are finite-dimensional, $\mathcal{V}_1^{\top} \bigotimes \cdots \bigotimes \mathcal{V}_n^{\top} \cong (\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n)^{\top}$ via the unique map $\Phi : \mathcal{V}_1^{\top} \bigotimes \cdots \bigotimes \mathcal{V}_n^{\top} \to (\mathcal{V}_1 \bigotimes \cdots \bigotimes \mathcal{V}_n)^{\top}$ for which

$$\Phi\left(\varphi_1\otimes\cdots\otimes\varphi_n\right)\left(v_1\otimes\cdots\otimes v_n\right)=\varphi_1\left(v_1\right)\cdots\varphi_n\left(v_n\right),$$

or, in other words,

$$\Phi\left(\varphi_1\otimes\cdots\otimes\varphi_n\right)=\varphi_1\cdots\varphi_n.$$

Thus, for example, if the \mathcal{V}_i are all equal to \mathcal{V} , and \mathcal{V} has the basis \mathcal{B} , then $\Phi\left(x_1^\top\otimes\cdots\otimes x_n^\top\right)=x_1^\top\cdots x_n^\top=\left(x_1\otimes\cdots\otimes x_n\right)^\top$, where each $x_i\in\mathcal{B}$.

5 Vector Algebras

5.1 The New Element: Vector Multiplication

A vector space per se has but one type of multiplication, namely multiplication (scaling) of vectors by scalars. However, additional structure may be added to a vector space by defining other multiplications. Defining a vector multiplication such that any two vectors may be multiplied to yield another vector, and such that this multiplication acts in a harmonious fashion with respect to vector addition and scaling, gives a new kind of structure when this vector multiplication is included as a structural element. The maps of such an enhanced structure, that is, the functions that preserve the entire structure, including the vector multiplication, are still vector space maps, but some vector space maps may no longer qualify due to the requirement that the vector multiplication must also be preserved. Adding a new structural element, giving whatever benefits it may, also then may give new burdens in the logical development of the theory involved if we are to exploit structural aspects through use of function images. We will see, though, that at the cost of a little added complication, some excellent constructs will be obtained as a result of including various vector multiplications.

Thus we will say that the vector space \mathcal{V} over the field \mathcal{F} becomes a **vector** algebra (over \mathcal{F}) when there is defined on it a **vector multiplication** $\mu: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, written $\mu(u, v) = u * v$, which is required to be bilinear so that it satisfies both the distributive laws

$$(t+u)*v = t*v + u*v$$
, and $u*(v+w) = u*v + u*w$

and also the law

$$a \cdot (u * v) = (a \cdot u) * v = u * (a \cdot v)$$

for all $a \in \mathcal{F}$ and all $t, u, v, w \in \mathcal{V}$. A vector algebra is **associative** or **commutative** according as its vector multiplication is associative or commutative. If there is a multiplicative neutral element (a **unit element**) the vector algebra is called **unital**. Since vector algebras are the only algebras treated here, we will often omit the "vector" qualifier and simply refer to a vector algebra as an **algebra**.

Exercise 5.1 v * 0 = 0 * v = 0.

Exercise 5.2 In a unital algebra with unit element 1, (-1) * v = -v.

A (vector) subalgebra of an algebra is a vector space subspace that is closed under the vector multiplication of the algebra, so that it is an algebra in its own right with a vector multiplication that has the same effect as the vector multiplication of the original algebra, but is actually its restriction to the subspace.

An **algebra map** (or sometimes simply **map** when the kind of map is clear) is a function between algebras over the same field, which function is a vector space map that also preserves vector multiplication. The image of a map is a subalgebra of the codomain and is associative (resp. commutative) when the domain algebra is. Under a map, the image of a unit element acts as a unit element in the image algebra.

Exercise 5.3 The inverse of a bijective algebra map is an algebra map. (The isomorphisms of algebras are thus the bijective algebra maps.)

5.2 Quotient Algebras

The **kernel** of an algebra map is defined to be the kernel of the same function viewed as a vector space map, namely those vectors of the domain that map to the zero vector of the codomain. The kernel of an algebra map is a subalgebra and more. It is an **ideal** in the algebra, i. e., a subalgebra closed under multiplication on the left, and on the right, by all the elements of the whole original algebra. Thus for an ideal \mathcal{I} in the algebra \mathcal{V} we have $x*v \in \mathcal{I}$ and $v*x \in \mathcal{I}$ for all $x \in \mathcal{I}$ and for all $v \in \mathcal{V}$. Every ideal is a subalgebra, but not every subalgebra is an ideal. Hence there can be subalgebras that are not kernels of any algebra map. However every ideal does turn out to be the kernel of some algebra map, as we shall soon see.

The level sets of an algebra map are the level sets of a vector space map, and they may be made into a vector space, and indeed into a vector algebra, in the same manner as was previously used in Section 2.2. This vector algebra made by mirroring the image of an algebra map in the level sets is the **quotient algebra** \mathcal{V}/\mathcal{K} of the vector algebra \mathcal{V} by the algebra map's kernel \mathcal{K} . The theme remains the same: a quotient "is" the image of a map, up to isomorphism, and all maps with the same domain and the same kernel have isomorphic images. In a quotient algebra, the product of cosets is **modular**, as the following exercise describes.

Exercise 5.4 Let V be a vector algebra and let K be the kernel of an algebra map from V. Then in V/K, (u + K) * (v + K) = u * v + K for all $u, v \in V$.

The cosets of any ideal make up a vector space quotient. It is always possible to impose the modular product on the cosets and thereby make the vector space quotient into an algebra.

Exercise 5.5 Let V be a vector algebra over \mathcal{F} and let \mathcal{I} be an ideal in V. On the vector space V/\mathcal{I} , prescribing the modular product

$$(u + \mathcal{I}) * (v + \mathcal{I}) = (u * v) + \mathcal{I}$$

for all $u, v \in \mathcal{V}$ gives a well-defined multiplication that makes \mathcal{V}/\mathcal{I} into a vector algebra over \mathcal{F} .

For any vector algebra $\mathcal V$ and ideal $\mathcal I \lhd \mathcal V$, the natural projection function $p: \mathcal V \to \mathcal V/\mathcal I$ that sends v to the coset $v+\mathcal I$ is a vector space map with kernel $\mathcal I$, according to Proposition 33. p is also an algebra map if we employ the modular product, for then $p(u)*p(v) = (u+\mathcal I)*(v+\mathcal I) = u*v+\mathcal I = p(u*v)$. Thus we have the following result.

Theorem 62 Let V be a vector algebra and let \mathcal{I} be an ideal in V. Then the natural map $p: V \to V/\mathcal{I}$ that sends v to the coset $v + \mathcal{I}$ is an algebra map with kernel \mathcal{I} , if the modular product is used on V/\mathcal{I} . Hence each ideal \mathcal{I} in V is the kernel of some algebra map, and thus a quotient algebra V/\mathcal{I} exists for every ideal \mathcal{I} in V.

5.3 Algebra Tensor Product

For vector algebras \mathcal{V} and \mathcal{W} over the field \mathcal{F} , the vector space tensor product $\mathcal{V} \bigotimes \mathcal{W}$ may be made into an algebra by requiring

$$(v \otimes w) * (v' \otimes w') = (v * v') \otimes (w * w')$$

and extending as a bilinear function on $(\mathcal{V} \otimes \mathcal{W}) \times (\mathcal{V} \otimes \mathcal{W})$ so that

$$\left(\sum_{i} v_{i} \otimes w_{i}\right) * \left(\sum_{j} v_{j}' \otimes w_{j}'\right) = \sum_{i,j} \left(v_{i} * v_{j}'\right) \otimes \left(w_{i} * w_{j}'\right).$$

This does give a well-defined vector multiplication for $\mathcal{V} \otimes \mathcal{W}$.

Proposition 63 Given vectors algebras \mathcal{V} and \mathcal{W} over the field \mathcal{F} , there exits a bilinear function $\mu: (\mathcal{V} \bigotimes \mathcal{W}) \times (\mathcal{V} \bigotimes \mathcal{W}) \to \mathcal{V} \bigotimes \mathcal{W}$ such that for all $v, v' \in \mathcal{V}$ and all $w, w' \in \mathcal{W}$, $\mu(v \otimes w, v' \otimes w') = (v * v') \otimes (w * w')$.

Proof: Consider the function from $\mathcal{V} \times \mathcal{W} \times \mathcal{V} \times \mathcal{W}$ to $\mathcal{V} \otimes \mathcal{W}$ that sends (v, w, v', w') to $(v * v') \otimes (w * w')$. This function is linear in each variable separately and therefore there is a vector space map from $\mathcal{V} \otimes \mathcal{W} \otimes \mathcal{V} \otimes \mathcal{W}$ to $\mathcal{V} \otimes \mathcal{W}$ that sends $v \otimes w \otimes v' \otimes w'$ to $(v * v') \otimes (w * w')$. Since there is an isomorphism from $(\mathcal{V} \otimes \mathcal{W}) \otimes (\mathcal{V} \otimes \mathcal{W})$ to $\mathcal{V} \otimes \mathcal{W} \otimes \mathcal{V} \otimes \mathcal{W}$ that sends $(v \otimes w) \otimes (v' \otimes w')$ to $v \otimes w \otimes v' \otimes w'$, there is therefore a vector space map from $(\mathcal{V} \otimes \mathcal{W}) \otimes (\mathcal{V} \otimes \mathcal{W})$ to $\mathcal{V} \otimes \mathcal{W}$ that sends $(v \otimes w) \otimes (v' \otimes w')$ to $(v * v') \otimes (w * w')$, and corresponding to this vector space map is a bilinear function $\mu : (\mathcal{V} \otimes \mathcal{W}) \times (\mathcal{V} \otimes \mathcal{W}) \to \mathcal{V} \otimes \mathcal{W}$ that sends $(v \otimes w, v' \otimes w')$ to $(v * v') \otimes (w * w')$. \blacksquare

 $\mathcal{V} \bigotimes \mathcal{W}$ with this vector multiplication function is the **algebra tensor product** of the vector algebras \mathcal{V} and \mathcal{W} over the field \mathcal{F} .

Exercise 5.6 If the vector algebras V and W over the field F are both commutative, or both associative, or both unital, then so is their algebra tensor product.

5.4 The Tensor Algebras of a Vector Space

The **contravariant tensor algebra** $\bigotimes \mathcal{V}$ of a vector space \mathcal{V} over the field \mathcal{F} is formed from the **tensor powers** $\bigotimes^0 \mathcal{V} = \mathcal{F}$, and $\bigotimes^k \mathcal{V} = \mathcal{V} \bigotimes \cdots \bigotimes \mathcal{V}$ (with k factors \mathcal{V} , $k = 1, 2, \ldots$), by taking the weak product $\biguplus_{k \geqslant 0} \bigotimes^k \mathcal{V}$ and defining on it a vector multiplication. To simplify notation, we will not distinguish $\bigotimes^k \mathcal{V}$ from its alias $\xi_k (\bigotimes \mathcal{V}) = \eta_k \left(\bigotimes^k \mathcal{V}\right)$ (using the functions introduced in section 3.7), so that, for example, $\bigotimes \mathcal{V}$ may also be written as the direct sum $\bigotimes \mathcal{V} = \bigoplus_{k \geqslant 0} \bigotimes^k \mathcal{V}$. For $u, v \in \bigotimes \mathcal{V}$ such that $u \in \bigotimes^i \mathcal{V}$ and $v \in \bigotimes^j \mathcal{V}$, u * v will be defined as the element of $\bigotimes^{i+j} \mathcal{V}$ that corresponds to $u \otimes v$ under the isomorphism of Theorem 58 or Theorem 61. In general, $u, v \in \bigotimes \mathcal{V} = \bigoplus_{k \geqslant 0} \bigotimes^k \mathcal{V}$ can be written uniquely as finite sums $u = \sum u_i$ and $v = \sum v_j$ where $u_i \in \bigotimes^i \mathcal{V}$ and $v_j \in \bigotimes^j \mathcal{V}$ and their product will then be defined as $u * v = \sum w_k$ where $w_k = \sum_{i+j=k} u_i * v_j$. With this bilinear vector multiplication, $\bigotimes \mathcal{V}$ is an associative, unital algebra. For most \mathcal{V} , this

algebra is not commutative, however. As is customary for this algebra, we will henceforth use \otimes instead of * as the vector multiplication symbol.

 $\bigotimes \mathcal{V}^{\top}$ is the **covariant tensor algebra** of a vector space \mathcal{V} . Taking the algebra tensor product of the contravariant algebra with the covariant algebra gives $\mathsf{T}\mathcal{V} = (\bigotimes \mathcal{V}) \bigotimes (\bigotimes \mathcal{V}^{\top})$, the **(full) tensor algebra** of \mathcal{V} . The elements of $\mathsf{T}\mathcal{V}$ are called **tensor combinations**. In $\mathsf{T}\mathcal{V}$, the algebra product of $r \in \bigotimes \mathcal{V}$ and $s \in \bigotimes \mathcal{V}^{\top}$ is denoted $r \otimes s$ and when the r is in $\bigotimes^{p} \mathcal{V}$ and the s is in $\bigotimes^{q} \mathcal{V}^{\top}$, the product $r \otimes s$ is said to be a **homogeneous tensor combination**, or simply a **tensor**, of **contravariant degree** p, of **covariant degree** q, and of **total degree** p+q. It is not hard to see that $\mathsf{T}\mathcal{V} = \bigoplus_{p,q \geqslant 0} \mathsf{T}_{q}^{p}\mathcal{V}$ where $\mathsf{T}_{q}^{p}\mathcal{V} = (\bigotimes^{p} \mathcal{V}) \bigotimes (\bigotimes^{q} \mathcal{V}^{\top})$. The elements of the form $v_{1} \otimes \cdots \otimes v_{p}$ in $\bigotimes^{p} \mathcal{V}$, p > 0, where each factor is

The elements of the form $v_1 \otimes \cdots \otimes v_p$ in $\bigotimes^p \mathcal{V}$, p > 0, where each factor is in $\bigotimes^1 \mathcal{V}$, are the **e-products** (**elementary products**) of **degree** p in $\bigotimes \mathcal{V}$. The nonzero elements of $\bigotimes^0 \mathcal{V}$ are the **e-products** of **degree** 0. $\bigotimes \mathcal{V}$ thus is the vector space of linear combinations of e-products of various degrees, and $T\mathcal{V}$ is the vector space of linear combinations of **mixed e-products** $r \otimes s$ where the various r and s are e-products of various degrees in $\bigotimes \mathcal{V}$ and $\bigotimes \mathcal{V}^\top$, respectively. This way of defining the tensor algebra of \mathcal{V} always keeps the elements of $\bigotimes^1 \mathcal{V}$ all together on the left, and the elements of $\bigotimes^1 \mathcal{V}^\top$ all together on the right, in each mixed e-product.

 $\bigotimes \mathcal{V}$ is a basic construct from which two important algebras may be derived as quotients, as we do in the next two sections.

5.5 The Exterior Algebra of a Vector Space

Given an ideal \mathcal{I} in an algebra \mathcal{W} , the quotient algebra \mathcal{W}/\mathcal{I} is a kind of scaled-down image of \mathcal{W} that suppresses \mathcal{I} and glues together into one image element all the elements of each coset $v + \mathcal{I}$ in \mathcal{W} , effectively treating the elements of \mathcal{W} modulo \mathcal{I} . Elements of a particular type in \mathcal{W} that can be expressed as the members of an ideal \mathcal{I} in \mathcal{W} can be "factored out" by passing to the quotient \mathcal{W}/\mathcal{I} . We now proceed along these lines by defining in $\bigotimes \mathcal{V}$ an ideal that when factored out will give us a very useful new algebra.

We say that the e-product $v_1 \otimes \cdots \otimes v_p$ is **dependent** if the sequence v_1, \ldots, v_p is dependent, that is if the set $\{v_1, \ldots, v_p\}$ is dependent or if there are any equal vectors among v_1, \ldots, v_p . We similarly call the tuple (v_1, \ldots, v_p) , **dependent** under the same circumstances. Such that are not dependent are **independent**. The set of values of all linear combinations of the set of dependent e-products in $\bigotimes \mathcal{V}$ will be denoted by \mathcal{D} . It is clear

that \mathcal{D} is an ideal in $\bigotimes \mathcal{V}$. The quotient algebra $\bigwedge \mathcal{V} = (\bigotimes \mathcal{V})/\mathcal{D}$ is the **exterior algebra** of \mathcal{V} . The **exterior e-product** $v_1 \wedge \cdots \wedge v_p$ is the image of the e-product $v_1 \otimes \cdots \otimes v_p$ under the natural projection $\pi_{\wedge} : \bigotimes \mathcal{V} \to \bigwedge \mathcal{V}$. $\bigwedge^p \mathcal{V} = \pi_{\wedge} (\bigotimes^p \mathcal{V})$, the pth **exterior power** of \mathcal{V} , is the subspace of elements of **degree** p, and is spanned by all the exterior e-products of degree p. Noting that $\bigwedge^0 \mathcal{V}$ is an alias of \mathcal{F} , and $\bigwedge^1 \mathcal{V}$ is an alias of \mathcal{V} , we will identify $\bigwedge^0 \mathcal{V}$ with \mathcal{F} and $\bigwedge^1 \mathcal{V}$ with \mathcal{V} . $\bigwedge \mathcal{V}$ is an associative algebra because it is the algebra map image of an associative algebra. It is customary to use \wedge as the multiplication symbol in $\bigwedge \mathcal{V}$.

Suppose that the set $\{v_1, \ldots, v_p\}$ of nonzero vectors in \mathcal{V} is dependent. Then for some $i, v_i = \sum_{i \neq i} a_i \cdot v_i$ and then

$$v_1 \otimes \cdots \otimes v_p = \sum_{j \neq i} a_j \cdot v_1 \otimes \cdots \otimes v_{i-1} \otimes v_j \otimes v_{i+1} \otimes \cdots \otimes v_p$$

so that such a dependent e-product is always a linear combination of e-products each of which has equal vectors among its factors. Hence the ideal \mathcal{D} in $\bigotimes \mathcal{V}$ may be more primitively described as the set of all linear combinations of e-products each of which has at least one pair of equal vectors among its factors.

Let $u, v \in \bigotimes^1 \mathcal{V}$. Then $0 = (u+v) \wedge (u+v) = u \wedge u + u \wedge v + v \wedge u + v \wedge v = u \wedge v + v \wedge u$ and therefore $u \wedge v = -(v \wedge u)$.

Exercise 5.7 Let $v_1 \wedge \cdots \wedge v_p$ be an exterior e-product. Then

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)} = (-1)^{\sigma} \cdot (v_1 \wedge \cdots \wedge v_p)$$

where $(-1)^{\sigma} = +1$ or -1 according as the permutation σ of $\{1, \ldots, p\}$ is even or odd.

Exercise 5.8 Let $r \in \bigwedge^p \mathcal{V}$ and let $s \in \bigwedge^q \mathcal{V}$. Then $s \wedge r = (-1)^{pq} \cdot r \wedge s$. (The sign changes only when both p and q are odd.)

Exercise 5.9 Let $r \in \bigwedge^p \mathcal{V}$. Then it follows at once from the above exercise that $r \wedge r = 0$ if p is odd and if $1 + 1 \neq 0$ in \mathcal{F} . What about when p is odd and 1 + 1 = 0 in \mathcal{F} ?

Let \mathcal{B} be a basis for \mathcal{V} . Based on \mathcal{B} , the set of **basis monomials** of degree p (all the e-products of degree p with factors chosen only from \mathcal{B}), is

a basis for $\bigotimes^p \mathcal{V}$. Let r denote the e-product $v_1 \otimes \cdots \otimes v_p$. Then r has an expansion of the form

$$r = \sum_{i_1} \cdots \sum_{i_p} a_{1,i_1} \cdots a_{p,i_p} \cdot x_{i_1} \otimes \cdots \otimes x_{i_p}$$

in terms of some $x_1, \ldots, x_N \in \mathcal{B}$. Let us write $r = r_+ r_+$, where r_- is the sum of those terms that have some equal subscripts and r_+ is the sum of the remaining terms that have no equal subscripts. We have

$$r_{\neq} = \sum_{i_1 < \dots < i_p} \sum_{\sigma \in \mathcal{S}_n} a_{1, i_{\sigma(1)}} \cdots a_{p, i_{\sigma(p)}} \cdot x_{i_{\sigma(1)}} \otimes \dots \otimes x_{i_{\sigma(p)}}$$

where S_p denotes the set of all permutations of $\{1, \ldots, p\}$. In the event that two of the factors, say v_l and v_m , l < m, are equal in r, the terms in the summation above occur in pairs with the same coefficient since

$$a_{1,i_{\sigma(1)}}\cdots a_{p,i_{\sigma(p)}}=a_{1,i_{\sigma(1)}}\cdots a_{l,i_{\sigma(m)}}\cdots a_{m,i_{\sigma(l)}}\cdots a_{p,i_{\sigma(p)}}.$$

When $v_l = v_m$, l < m, we may thus write

$$r_{\neq} = \sum_{i_1 < \dots < i_p} \sum_{\sigma \in \mathcal{A}_p} a_{1, i_{\sigma(1)}} \cdots a_{p, i_{\sigma(p)}} \cdot$$

$$\cdot (x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(p)}} + x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(m)}} \otimes \cdots \otimes x_{i_{\sigma(l)}} \otimes \cdots \otimes x_{i_{\sigma(p)}})$$

where \mathcal{A}_p is the set of all *even* permutations of $\{1,\ldots,p\}$. We say that the two e-products $v_1 \otimes \cdots \otimes v_p$ and $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}$ are of the **same parity**, or of **opposite parity**, according as σ is an even, or an odd, permutation of $\{1,\ldots,p\}$. We see therefore that the dependent basis monomials, along with the sums of pairs of independent basis monomials of opposite parity, span \mathcal{D} .

Let $\mathcal{X} = \{x_1, \ldots, x_p\}$ be a subset of the basis \mathcal{B} for \mathcal{V} . From the elements of \mathcal{X} , K = p! independent basis monomials may be formed by multiplying together the elements of \mathcal{X} in the various possible orders. Let $\mathcal{T}_1 = \{t_1, t_3, \ldots, t_{K-1}\}$ be the independent basis monomials of degree p, with factors from \mathcal{X} , and of the same parity as $t_1 = x_1 \otimes \cdots \otimes x_p$, and let $\mathcal{T}_2 = \{t_2, t_4, \ldots, t_K\}$ be those of the opposite parity. Then

$$\mathcal{T} = \{t_1, t_1 + t_2, t_2 + t_3, \dots, t_{K-1} + t_K\}$$

is a set of K independent elements with the same span as $\mathcal{T}_1 \cup \mathcal{T}_2$. Moreover, we claim that for any $s \in \mathcal{T}_1$ and any $t \in \mathcal{T}_2$, s + t is in the span of $\mathcal{T} \setminus \{t_1\}$.

It suffices to show this for s+t of the form t_i+t_k where i < k and exactly one of i, k is odd, or, what is the same, for $s+t=t_i+t_{i+2j+1}$. But $t_i+t_{i+2j+1}=(t_i+t_{i+1})-(t_{i+1}+t_{i+2})+(t_{i+2}+t_{i+3})-\cdots+(t_{i+2j}+t_{i+2j+1})$, verifying the claim. The following result is now clear.

Proposition 64 Let \mathcal{B} be a basis for \mathcal{V} . From each nonempty subset $\mathcal{X} = \{x_1, \ldots, x_p\}$ of p elements of \mathcal{B} , let the sets $\mathcal{T}_1 = \{t_1, t_3, \ldots, t_{K-1}\}$ and $\mathcal{T}_2 = \{t_2, t_4, \ldots, t_K\}$ comprising in total the K = p! degree p independent basis monomials be formed, \mathcal{T}_1 being those that are of the same parity as $t_1 = x_1 \otimes \cdots \otimes x_p$, and \mathcal{T}_2 being those of parity opposite to t_1 , and let

$$\mathcal{T} = \{t_1, t_1 + t_2, t_2 + t_3, \dots, t_{K-1} + t_K\}.$$

Then $\langle \mathcal{T} \rangle = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$ and if $s \in \mathcal{T}_1$ and $t \in \mathcal{T}_2$, then $s + t \in \langle \mathcal{T} \setminus \{t_1\} \rangle$. Let \mathcal{A}_0 denote the set of all dependent basis monomials based on \mathcal{B} , let \mathcal{A}_1 denote the union of all the sets $\mathcal{T} \setminus \{t_1\}$ for all p, and let \mathcal{E} denote the union of all the singleton sets $\{t_1\}$ for all p, and $\{1\}$. Then $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{E}$ is a basis for $\bigotimes \mathcal{V}$, $\mathcal{A}_0 \cup \mathcal{A}_1$ is a basis for the ideal \mathcal{D} of all linear combinations of dependent e-products in $\bigotimes \mathcal{V}$, and \mathcal{E} is a basis for a complementary subspace of \mathcal{D} .

Inasmuch as any independent set of vectors is part of some basis, in the light of Proposition 31 we then immediately infer the following useful result which also assures us that in suppressing the dependent e-products we have not suppressed any independent ones.

Corollary 65 (Criterion for Independence) The sequence v_1, \ldots, v_p of vectors of \mathcal{V} is independent if and only if $v_1 \wedge \cdots \wedge v_p \neq 0$.

Exercise 5.10 Let \mathcal{V} be an n-dimensional vector space with basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. Then for $1 \leq p \leq n$, the set $\{x_{i_1} \wedge \cdots \wedge x_{i_p} | i_1 < \cdots < i_p\}$ of $\binom{n}{p}$ elements is a basis for $\bigwedge^p \mathcal{V}$, while for p > n, $\bigwedge^p \mathcal{V} = 0$. In particular, then, the singleton set $\{x_1 \wedge \cdots \wedge x_n\}$ is a basis for $\bigwedge^n \mathcal{V}$.

We may restrict the domain of the natural projection $\pi_{\wedge}: \bigotimes \mathcal{V} \to \bigwedge \mathcal{V}$ to $\bigotimes^p \mathcal{V}$ and the codomain to $\bigwedge^p \mathcal{V}$ and thereby obtain the vector space map $\pi_{\wedge^p}: \bigotimes^p \mathcal{V} \to \bigwedge^p \mathcal{V}$ which has kernel $\mathcal{D}_p = \mathcal{D} \cap \bigotimes^p \mathcal{V}$. It is easy to see that $\bigwedge^p \mathcal{V} = \bigotimes^p \mathcal{V}/\mathcal{D}_p$ so that π_{\wedge^p} is the natural projection of $\bigwedge^p \mathcal{V}$ onto $\bigotimes^p \mathcal{V}/\mathcal{D}_p$. Let $\Upsilon_p: \mathcal{V}^p \to \bigotimes^p \mathcal{V}$ be a tensor product function, and let f be a

p-linear function from \mathcal{V}^p to a vector space \mathcal{W} over the same field as \mathcal{V} . Then by the universal property of the tensor product, there is a unique vector space map $f_{\otimes}: \bigotimes^p \mathcal{V} \to \mathcal{W}$ such that $f = f_{\otimes} \circ \Upsilon_p$. Assume now that f is zero on dependent p-tuples, so that the corresponding f_{\otimes} is zero on dependent p-tuples of degree p. Applying Theorem 34, we infer the existence of a unique map $f_{\wedge}: \bigwedge^p \mathcal{V} \to \mathcal{W}$ such that $f_{\otimes} = f_{\wedge} \circ \pi_{\wedge^p}$. Thus there is a universal property for the p-th exterior power, which we now officially record as a theorem. (An **alternating** p-linear function is one that vanishes on dependent p-tuples.)

Theorem 66 (Universal Property of Exterior Powers) Let V and W be vector spaces over the same field, and let $f: V^p \to W$ be an alternating p-linear function. Then there is a unique vector space map $f_{\wedge}: \bigwedge^p V \to W$ such that

$$f_{\wedge}(v_1 \wedge \cdots \wedge v_p) = f(v_1, \dots, v_p).$$

Exercise 5.11 Let $f: \mathcal{V} \to \mathcal{W}$ be a vector space map. Then there is a unique map $\bigwedge^p f: \bigwedge^p \mathcal{V} \to \bigwedge^p \mathcal{W}$ that satisfies

$$\bigwedge^{p} f(v_{1} \wedge \cdots \wedge v_{p}) = f(v_{1}) \wedge \cdots \wedge f(v_{p}).$$

The map $\bigwedge^p f$ of the above exercise is commonly called the pth **exterior** power of f.

Let $f: \mathcal{V} \to \mathcal{V}$ be a vector space map from the *n*-dimensional vector space \mathcal{V} to itself. Then by the above exercise there is a unique map $\bigwedge^n f: \bigwedge^n \mathcal{V} \to \bigwedge^n \mathcal{V}$ such that $\bigwedge^n f(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$. Since $\bigwedge^n \mathcal{V}$ is 1-dimensional, $\bigwedge^n f(t) = a \cdot t$ for some scalar $a = \det f$, the **determinant** of f. Note that the determinant of f is independent of any basis choice. However, the determinant is only defined for self-maps on finite-dimensional spaces.

Theorem 67 (Product Theorem) Let f, g be vector space maps on an n-dimensional vector space. Then $\det g \circ f = (\det g)(\det f)$.

Proof:
$$\bigwedge^n (g \circ f) (v_1 \wedge \cdots \wedge v_n) = g(f(v_1)) \wedge \cdots \wedge g(f(v_n)) = (\det g) \cdot f(v_1) \wedge \cdots \wedge f(v_n) = (\det g) (\det f) \cdot v_1 \wedge \cdots \wedge v_n$$
.

Proposition 68 Let $f: \mathcal{V} \to \mathcal{W}$ be a vector space map from the n-dimensional vector space \mathcal{V} to its alias \mathcal{W} . Let $\{x_1, \ldots, x_n\}$ be a basis for \mathcal{V} . Then f is invertible if and only if $f(x_1) \wedge \cdots \wedge f(x_n) \neq 0$.

Proof: Suppose that $f(x_1) \wedge \cdots \wedge f(x_n) = 0$. Then, by Corollary 65, the sequence $f(x_1), \ldots, f(x_n)$ is dependent, so there exist scalars a_1, \ldots, a_n , not all zero, such that

$$0 = a_1 \cdot f(x_1) + \ldots + a_n \cdot f(x_n) = f(a_1 \cdot x_1 + \ldots + a_n \cdot x_n).$$

Hence f sends the nonzero vector $a_1 \cdot x_1 + \ldots + a_n \cdot x_n$ to 0, and thus the kernel of f is not $\{0\}$. Therefore f fails to be one-to-one and hence is not invertible.

On the other hand, if $f(x_1) \wedge \cdots \wedge f(x_n) \neq 0$, the sequence $f(x_1), \ldots, f(x_n)$ is independent and makes up a basis of \mathcal{W} . Thus f sends each nonzero vector in \mathcal{V} to a nonzero vector in \mathcal{W} , and thus the kernel of f is $\{0\}$. Therefore f is one-to-one and hence invertible.

Corollary 69 Let $f: \mathcal{V} \to \mathcal{V}$ be a vector space map from the n-dimensional vector space \mathcal{V} to itself. Then f is invertible if and only if $\det f \neq 0$.

Suppose the *n*-dimensional vector space aliases \mathcal{V} and \mathcal{W} have the respective bases $\{x_1, \ldots, x_n\}$, and $\{y_1, \ldots, y_n\}$. The map $f: \mathcal{V} \to \mathcal{W}$ then sends x_j to $\sum_i a_{i,j} \cdot y_i$ for some scalars $a_{i,j}$. We have

$$f(x_1) \wedge \cdots \wedge f(x_n) = \left(\sum_i a_{i,1} \cdot y_i\right) \wedge \cdots \wedge \left(\sum_i a_{i,n} \cdot y_i\right) =$$

$$= \sum_{\sigma \in \mathcal{S}_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \cdot y_{\sigma(1)} \wedge \cdots \wedge y_{\sigma(n)} =$$

$$= \left(\sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n}\right) \cdot y_1 \wedge \cdots \wedge y_n$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$ and $(-1)^{\sigma} = +1$ or -1 according as the permutation σ is even or odd. Therefore f is invertible if and only if

$$0 \neq \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n} = \det \left[a_{i,j} \right],$$

i. e., if and only if the familiar determinant of the $n \times n$ matrix $[a_{i,j}]$ is nonzero. We observe that when $\mathcal{V} = \mathcal{W}$ and $x_i = y_i$ for all i, det $[a_{i,j}] = \det f$. We record this result as follows.

Proposition 70 Let the n-dimensional vector space aliases \mathcal{V} and \mathcal{W} have the respective bases $\{x_1, \ldots, x_n\}$, and $\{y_1, \ldots, y_n\}$. Let the map $f: \mathcal{V} \to \mathcal{W}$ send each x_j to $\sum_i a_{i,j} \cdot y_i$. Then f is invertible if and only if the determinant of the matrix $[a_{i,j}]$, denoted $\det[a_{i,j}]$, is nonzero. When $\mathcal{V} = \mathcal{W}$ and $x_i = y_i$ for all i, $\det[a_{i,j}] = \det f$.

Exercise 5.12 The determinant of the $n \times n$ matrix $[a_{i,j}]$ is an alternating n-linear functional of its columns (and also of its rows, due to the familiar result that a matrix and its transpose have the same determinant).

The universal property of exterior powers leads at once to the conclusion that if \mathcal{V} is over the field \mathcal{F} , the map space subspace $\left\{\mathcal{V}^{p} \stackrel{(alt \ p-lin)}{\rightarrow} \mathcal{F}\right\}$ of alternating p-linear functionals is isomorphic to $(\bigwedge^p \mathcal{V})^{\top}$. There is also an analog of Theorem 54, which we now start to develop by exploring the coordinate functions on $\bigwedge^p \mathcal{V}$ relative to a basis \mathcal{B} for \mathcal{V} . Let $t = v_1 \wedge \cdots \wedge v_p$ be an exterior e-product in $\bigwedge^p \mathcal{V}$. Let $x_1 \wedge \cdots \wedge x_p$ be a typical basis vector of $\bigwedge^p \mathcal{V}$, formed from elements $x_i \in \mathcal{B}$, and let t be expanded in terms of such basis vectors. Then the coordinate function $(x_1 \wedge \cdots \wedge x_p)^{\top}$ corresponding to $x_1 \wedge \cdots \wedge x_p$ is the element of $(\bigwedge^p \mathcal{V})^{\top}$ that gives the coefficient of $x_1 \wedge \cdots \wedge x_p$ in this expansion of t. This coefficient is readily seen to be det $[x_i^{\top}(v_i)]$, where x_i^{\top} is the coordinate function on \mathcal{V} that corresponds to $x_i \in \mathcal{B}$. Thus we are led to consider what are easily seen to be alternating p-linear functionals f_{ϕ_1,\dots,ϕ_p} of the form $f_{\phi_1,...,\phi_p}(v_1,...,v_p) = \det [\phi_i(v_j)]$ where the ϕ_i are linear functionals on \mathcal{V} . By the universal property, $f_{\phi_1,\ldots,\phi_p}(v_1,\ldots,v_p) = \widehat{f}_{\phi_1,\ldots,\phi_p}(v_1\wedge\cdots\wedge v_p)$ for a unique linear functional $\widehat{f}_{\phi_1,\dots,\phi_p} \in (\bigwedge^p \mathcal{V})^{\top}$. Hence there is a function $\Phi: (\mathcal{V}^{\top})^p \to (\bigwedge^p \mathcal{V})^{\top}$ such that $\Phi(\phi_1, \dots, \phi_p) = \widehat{f}_{\phi_1, \dots, \phi_p}$. But Φ is clearly an alternating p-linear function on $(\mathcal{V}^{\top})^p$ and by the universal property there is a unique vector space map $\widehat{\Phi}: \bigwedge^p \mathcal{V}^\top \to (\bigwedge^p \mathcal{V})^\top$ such that $\widehat{\Phi}(\phi_1 \wedge \cdots \wedge \phi_p) = \Phi(\phi_1, \dots, \phi_p)$. Based on our study of coordinate functions above, we see that $\widehat{\Phi}(x_1^\top \wedge \cdots \wedge x_p^\top) = (x_1 \wedge \cdots \wedge x_p)^\top$. Therefore, for finite-dimensional \mathcal{V} , $\widehat{\Phi}$ is an isomorphism because it sends a basis to a basis which makes it onto, and it is therefore also one-to-one by Theorem 25. The following statement summarizes our results.

Theorem 71 Let \mathcal{V} be a vector space over the field \mathcal{F} . Then $(\bigwedge^p \mathcal{V})^{\top} \cong \{\mathcal{V}^p \overset{(alt\ p-lin)}{\to} \mathcal{F}\}$. If \mathcal{V} is finite-dimensional, we also have $\bigwedge^p \mathcal{V}^{\top} \cong (\bigwedge^p \mathcal{V})^{\top}$ via the map $\widehat{\Phi} : \bigwedge^p \mathcal{V}^{\top} \to (\bigwedge^p \mathcal{V})^{\top}$ for which $\widehat{\Phi} (\phi_1 \wedge \cdots \wedge \phi_p) (v_1 \wedge \cdots \wedge v_p) = \det [\phi_i(v_j)]$.

Exercise 5.13 What is $\widehat{\Phi}$ in the case p = 1?

Exterior algebra is an important subject that will receive further attention in later chapters. Now we consider another example of a vector algebra that is an algebra map image of the contravariant tensor algebra $\bigotimes \mathcal{V}$.

5.6 The Symmetric Algebra of a Vector Space

Noncommutativity in $\bigotimes \mathcal{V}$ can be suppressed by passing to the quotient $SV = \bigotimes V/N$ where N is an ideal that expresses noncommutativity. The result is a commutative algebra that is customarily called the **symmetric** algebra of \mathcal{V} . The noncommutativity ideal \mathcal{N} may be taken to be the set of all linear combinations of differences of pairs of e-products of the same degree $p, p \ge 2$, which contain the same factors with the same multiplicities, but in different orders. The effect of passing to the quotient then is to identify all the e-products of the same degree which have the same factors with the same multiplicaties without regard to the order in which these factors are being multiplied. The symmetric e-product $v_1 \cdots v_p$ is the image of the e-product $v_1 \otimes \cdots \otimes v_p$ under the natural projection $\pi_S : \bigotimes \mathcal{V} \to S\mathcal{V}$. $S^p\mathcal{V} =$ $\pi_{S}(\bigotimes^{p} \mathcal{V})$, the pth symmetric power of \mathcal{V} , is the subspace of elements of **degree** p. Noting that S^0V is an alias of \mathcal{F} , and S^1V is an alias of \mathcal{V} , we will identify $S^0\mathcal{V}$ with \mathcal{F} and $S^1\mathcal{V}$ with \mathcal{V} . We will write products in $S\mathcal{V}$ with no sign to indicate the product operation, and we will indicate repeated adjacent factors by the use of an exponent. Thus $r_3r_2^2r_1$, $r_2r_3r_1r_2$, and $r_1r_2^2r_3$ are typical (and equal) product expressions in SV.

Following a plan similar to that used with the exterior algebra, we will produce a basis for the ideal \mathcal{N} and a basis for a complementary subspace of \mathcal{N} . Let \mathcal{B} be a basis for \mathcal{V} . Consider the difference of the pair of e-products $v_1 \otimes \cdots \otimes v_p$ and $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}$, where σ is some permutation of $\{1, \ldots, p\}$. In terms of some $x_1, \ldots, x_N \in \mathcal{B}$ we have for this difference

$$\sum_{i_1} \cdots \sum_{i_p} (a_{1,i_1} \cdots a_{p,i_p} \cdot x_{i_1} \otimes \cdots \otimes x_{i_p} - a_{\sigma(1),i_{\sigma(1)}} \cdots a_{\sigma(p),i_{\sigma(p)}} \cdot x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(p)}})$$

$$= \sum_{i_1} \cdots \sum_{i_p} a_{1,i_1} \cdots a_{p,i_p} \cdot (x_{i_1} \otimes \cdots \otimes x_{i_p} - x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(p)}}),$$

where the multiplicative commutativity of the scalars has been exploited. Thus \mathcal{N} is spanned by the differences of pairs of basis monomials that contain the same factors with the same multiplicities, but in different orders.

To each basis monomial $t = x_1 \otimes \cdots \otimes x_p$, where the x_i are (not necessarily distinct) elements of \mathcal{B} , there corresponds the **multiset**

$$\mathcal{M} = \{(\xi_1, \nu_1), \dots, (\xi_m, \nu_m)\}\$$

where the ξ_j are the distinct elements of $\{x_1, \ldots, x_p\}$ and ν_j is the multiplicity with which ξ_j appears as a factor in t. Putting $\nu = (\nu_1, \ldots, \nu_m)$ and $|\nu| = \nu_1 + \cdots + \nu_m$, we then have $|\nu| = p$. Given a particular multiset \mathcal{M} , let $\mathcal{T}_0 = \{t_1, \ldots, t_K\}$ be the set of all basis monomials to which \mathcal{M} corresponds. Here $K = \binom{|\nu|}{\nu}$ is the multinomial coefficient given by $\binom{|\nu|}{\nu} = |\nu|!/\nu!$ where $\nu! = (\nu_1!)\cdots(\nu_m!)$. From each such \mathcal{T}_0 we may form the related set $\mathcal{T} = \{t_1, t_1 - t_2, \ldots, t_{K-1} - t_K\}$ with the same span. Now $t_i - t_{i+j} = (t_i - t_{i+1}) + \cdots + (t_{i+j-1} - t_{i+j})$ so that the difference of any two elements of \mathcal{T}_0 is in $\langle \mathcal{T} \setminus \{t_1\} \rangle$. We therefore have the following result.

Proposition 72 Let \mathcal{B} be a basis for \mathcal{V} . From each nonempty multiset $\mathcal{M} = \{(\xi_1, \nu_1), \ldots, (\xi_k, \nu_m)\}$ where the ξ_j are distinct elements of \mathcal{B} , let the set $\mathcal{T}_0 = \{t_1, \ldots, t_K\}$ of basis monomials that correspond be formed, where $K = \binom{|\nu|}{\nu}$. Let

$$\mathcal{T} = \{t_1, t_1 - t_2, t_2 - t_3, \dots, t_{K-1} - t_K\}.$$

Then $\langle \mathcal{T} \rangle = \langle \mathcal{T}_0 \rangle$ and if $s, t \in \mathcal{T}_0$, then $s - t \in \langle \mathcal{T} \setminus \{t_1\} \rangle$. Let \mathcal{A} denote the union of all the sets $\mathcal{T} \setminus \{t_1\}$ for all ν , and let \mathcal{S} denote the union of all the singleton sets $\{t_1\}$ for all ν , and $\{1\}$. Then $\mathcal{A} \cup \mathcal{S}$ is a basis for $\bigotimes \mathcal{V}$, \mathcal{A} is a basis for the noncommutativity ideal \mathcal{N} , and \mathcal{S} is a basis for a complementary subspace of \mathcal{N} .

Exercise 5.14 Let V be an n-dimensional vector space with basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. Then for p > 0, the set $\{x_{i_1} \cdots x_{i_p} \mid i_1 \leqslant \cdots \leqslant i_p\}$ of $\binom{n+p-1}{p}$ elements is a basis for S^pV .

Exercise 5.15 (Universal Property of Symmetric Powers) Let V and W be vector spaces over the same field, and let $f: V^p \to W$ be a symmetric p-linear function, i. e., a p-linear function such that $f(v_1, \ldots, v_p) = f\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right)$ for every permutation σ of $\{1, \ldots, p\}$. Then there is a unique vector space map $f_S: S^pV \to W$ such that $f(v_1, \ldots, v_p) = f_S(v_1 \cdots v_p)$.

Exercise 5.16 $(S^p \mathcal{V})^{\top}$ is isomorphic to the vector space $\{\mathcal{V}^p \overset{(sym \ p-lin)}{\rightarrow} \mathcal{F}\}$ of symmetric p-linear functionals.

5.7 Null Pairs and Cancellation

By a **null pair** in an algebra we mean a pair u, v of nonzero elements, not necessarily distinct, such that u * v = 0. (We avoid the rather misleading term zero divisor, or the similar divisor of zero, usually used in describing this notion.) It is the same for an algebra to have no null pair as it is for it to support cancellation of nonzero factors, as the following proposition details.

Proposition 73 An algebra has no null pair if and only if it supports **cancellation**: the equations u * x = u * y and x * v = y * v, where u, v are any nonzero elements, each imply x = y.

Proof: Suppose that the algebra has no null pair. The equation u*x = u*y with $u \neq 0$ implies that u*(x-y) = 0 which then implies that x-y = 0 since the algebra has no null pair. Similarly x*v = y*v with $v \neq 0$ implies x-y = 0. Thus in each case, x = y as was to be shown.

On the other hand, assume that cancellation is supported. Suppose that u * v = 0 with $u \neq 0$. Then u * v = u * 0, and canceling u gives v = 0. Hence u * v = 0 with $u \neq 0$ implies that v = 0 and it is therefore not possible that u * v = 0 with both u and v nonzero.

An element u is called **left regular** if there is no nonzero element v such that u * v = 0, and similarly v is called **right regular** if there is no nonzero u such that u * v = 0. An element is **regular** if it is both left regular and right regular.

Exercise 5.17 u may be canceled from the left of u * x = u * y if and only if it is left regular, and v may be canceled from the right of x * v = y * v if and only if it is right regular.

5.8 Problems

- 1. Give an example of an algebra map $f: A \to B$ where A and B are both unital with unit elements 1_A and 1_B respectively, but $f(1_A) \neq 1_B$.
- 2. Use exterior algebra to derive the *Laplace expansion* of the determinant of an $n \times n$ matrix.

- 3. Use exterior algebra to derive $Cramer's \ rule$ for solving a set of n linear equations in n unknowns when the coefficient matrix has nonzero determinant. Then derive the well-known formula for the inverse of the coefficient matrix in terms of cofactors.
- 4. If \mathcal{V} is finite-dimensional over a field of *characteristic* 0, then $S^p \mathcal{V}^{\top} \cong (S^p \mathcal{V})^{\top}$ via the vector space map $\widehat{\Psi} : S^p \mathcal{V}^{\top} \to (S^p \mathcal{V})^{\top}$ for which $\widehat{\Psi} (\psi_1 \cdots \psi_p) (r_1 \cdots r_p) = \operatorname{per} [\psi_i (r_j)]$ where $\operatorname{per} [a_{i,j}] = \sum_{\sigma \in \mathcal{S}_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n}$ is the *permanent* of the matrix $[a_{i,j}]$.
- 5. If u * v = 0 in an algebra, must then v * u = 0 also?
- 6. An algebra supports **left-cancellation** if v * x = v * y implies x = y whenever $v \neq 0$, and **right-cancellation** is defined similarly. An algebra supports left-cancellation if and only if it supports right-cancellation.

6 Vector Affine Geometry

6.1 Basing a Geometry on a Vector Space

One may base a geometry on a vector space by defining the fundamental geometric objects to be the members of some specified family of subsets of vectors, and by declaring that two such objects are *incident* if one is included in the other by set inclusion. It is natural to take as fundamental geometric objects a collection of "flat" sets, such as subspaces or cosets of subspaces, and this is what we shall do here. By basing a geometry in the same way on each vector space of a family of vector spaces over the same field, we will, upon also specifying suitable structure-preserving maps, be able to view the entire collection of fundamental geometric objects of all the vector spaces of the family as a *category* of geometric spaces having the same kind of structure.

In this chapter we derive from an underlying vector space over a general field a type of geometry called *affine*, which embodies the more primitive notions of what is no doubt the most familiar type of geometry, namely Euclidean geometry. In affine geometry, just like in Euclidean geometry, there is uniformity in the sense that the view from each point is the same. The points all have equal standing, and any one of them may be considered to be the origin. In addition to incidence, we also have parallelism as a fundamental concept in affine geometry. Working, as we will, over a general, possibly unordered, field, the availability of geometrically useful quantities that can be expressed as field values will be quite limited, and even over the real numbers, far fewer such quantities are available to us than in ordinary Euclidean geometry. Nevertheless, many important concepts and results are embraced within affine geometry, and it provides a suitable starting point for our exploration of vector geometry.

6.2 Affine Flats

Let \mathcal{V} be a vector space \mathcal{V} over the field \mathcal{F} . We denote by $V(\mathcal{V})$ the set of all subspaces of \mathcal{V} . We utilize $V(\mathcal{V})$ to obtain an affine geometry by taking as fundamental geometric objects the set $\mathcal{V}+V(\mathcal{V})$ of all **affine flats** of \mathcal{V} , which are just the cosets of the vector space subspaces of \mathcal{V} . Thus we introduce $A(\mathcal{V}) = \mathcal{V} + V(\mathcal{V})$ as the **affine structure** on \mathcal{V} . In $A(\mathcal{V})$, the **points** are the cosets of $\{0\}$ (i. e., the singleton subsets of \mathcal{V} , or essentially the

vectors of \mathcal{V}), the **lines** are the cosets of the one-dimensional subspaces, the **planes** are the cosets of the two-dimensional subspaces, and the n-planes are the cosets of the n-dimensional subspaces. The **hyperplanes** are the cosets of the subspaces of codimension 1. The **dimension** of an affine flat is the dimension of the subspace of which it is a coset. If $\Phi \in A(\mathcal{V})$ is given by $\Phi = v + \mathcal{W}$ where $\mathcal{W} \in V(\mathcal{V})$, we call \mathcal{W} the **directional subspace**, or **underlying subspace**, of Φ . The notation Φ^{v} will be used to signify the unique directional subspace of the affine flat Φ .

6.3 Parallelism in Affine Structures

We say that two affine flats, distinct or not, in the same affine structure are **parallel** if their directional subspaces are incident (one is contained entirely within the other). They are **strictly parallel** if their directional subspaces are equal. They are **nontrivially parallel** if they are parallel and neither is a point. They are **properly parallel** if they are parallel and unequal. Any of these terms applies to a set of affine flats if it applies pairwise within the set. Thus, for example, to say that S is a set of strictly parallel affine flats means that for every $\Phi, \Psi \in S$, Φ and Ψ are strictly parallel.

Exercise 6.1 Two strictly parallel affine flats that intersect must be equal. Hence, two parallel affine flats that intersect must be incident.

Exercise 6.2 The relation of being strictly parallel is an equivalence relation. However, in general, the relation of being parallel is not transitive.

6.4 Translates and Other Expressions

For vectors v, w, x and scalar a, a **translate** of v (in the direction of x - w) is an expression of the form $v + a \cdot (x - w)$. This is a fundamental affine concept. If $x, w \in \mathcal{S}$, we call $v + a \cdot (x - w)$ an \mathcal{S} -translate of v.

Proposition 74 The nonempty subset S of the vector space V is an affine flat if and only if it contains the value of each S-translate of each of its elements.

Proof: Let $S \subset V$ be such that it contains each S-translate of each of its elements and let $v \in S$. Then we wish to show that S = v + W where $W \triangleleft V$. This is the same as showing that 1) $v + a \cdot (w - v) \in S$ for all $w \in S$ and all scalars a, and that 2) $v + ((w - v) + (x - v)) \in S$ for all $w, x \in S$. The expression in 1) is clearly an S-translate of v and hence is in S as required. The expression in 2) may be written as v + ((w + (x - v)) - v) and is seen to be in S because z = w + (x - v) is an S-translate of w and is therefore in S so that our expression v + (z - v) is also an S-translate of v and must be in S.

On the other hand, suppose that S is an affine flat in V and let $v \in S$. Let $x = v + x^{\mathsf{v}}$ and $w = v + w^{\mathsf{v}}$ where $x^{\mathsf{v}}, w^{\mathsf{v}} \in S^{\mathsf{v}}$. Then $a \cdot (x - w) = a \cdot (x^{\mathsf{v}} - w^{\mathsf{v}}) \in S^{\mathsf{v}}$ and hence $v + a \cdot (x - w) \in S$.

Exercise 6.3 The intersection of affine flats is an affine flat or empty.

Iterating the process of forming translates gives the multi-translate

$$v + b_1 \cdot (x_1 - w_1) + \cdots + b_n \cdot (x_n - w_n)$$

of the vector v in the directions of the $x_i - w_i$ based on the vectors w_i, x_i and the scalars b_i .

An expression of the form $a_1 \cdot v_1 + \cdots + a_n \cdot v_n$ is a **linear expression** in the vectors v_1, \ldots, v_n , and $a_1 + \cdots + a_n$ is its **weight**. Two important cases are specially designated: A linear expression of weight 1 is an **affine expression**, and a linear expression of weight 0 is a **directional expression**.

If we expand out each of the terms $b_i \cdot (x_i - w_i)$ as $b_i \cdot x_i + (-b_i) \cdot w_i$, the multi-translate above becomes an affine expression in the w_i , the x_i , and v, and it becomes v plus a directional expression in the w_i and the x_i . On the other hand, since

$$a_1 \cdot v_1 + \dots + a_n \cdot v_n = (a_1 + \dots + a_n) \cdot v_1 + a_2 \cdot (v_2 - v_1) + \dots + a_n \cdot (v_n - v_1)$$

our typical affine expression may be written as the multi-translate

$$v_1 + a_2 \cdot (v_2 - v_1) + \cdots + a_n \cdot (v_n - v_1),$$

and our typical directional expression may be written as the linear expression

$$a_2 \cdot (v_2 - v_1) + \cdots + a_n \cdot (v_n - v_1)$$

in the differences $(v_2 - v_1), \ldots, (v_n - v_1)$.

Exercise 6.4 A directional expression in the vectors v_1, \ldots, v_n is nontrivially equal to 0 (some scalar coefficient not 0) if and only if some v_j is equal to an affine expression in the other v_i .

6.5 Affine Span and Affine Sum

An **affine combination** of a finite nonempty set of vectors is any linear combination of the set such that the coefficients sum to 1, and thus is an affine expression equal to a multi-translate of any one of the vectors based on all of them. If \mathcal{X} is a nonempty set of vectors, its **affine span** $\langle \mathcal{X} \rangle_A$ is the set of all affine combinations of all its finite nonempty subsets. We say that \mathcal{X} **affine-spans** $\langle \mathcal{X} \rangle_A$, and is an **affine spanning set** for $\langle \mathcal{X} \rangle_A$.

Exercise 6.5 Let \mathcal{X} be a nonempty subset of the vector space \mathcal{V} . Then $\langle \mathcal{X} \rangle_{\mathsf{A}}$ is the smallest affine flat containing \mathcal{X} , i. e., it is the intersection of all affine flats containing \mathcal{X} .

Let $\Phi, \Psi \in A(\mathcal{V})$. Then $\Phi +_A \Psi = \langle \Phi \cup \Psi \rangle_A$ is their **affine sum**, and similarly for more summands. With partial order \subset , meet \cap , and join $+_A$, $A(\mathcal{V}) \cup \{\emptyset\}$ is a *complete lattice* which is *upper semi-modular* but in general is neither *modular* nor *distributive* (see, e. g., Hall's *The Theory of Groups* for definitions and characterizations).

Exercise 6.6 Let $\Phi, \Psi \in A(\mathcal{V})$. Suppose first that $\Phi \cap \Psi \neq \emptyset$. Then $(\Phi +_A \Psi)^{\mathsf{v}} = \Phi^{\mathsf{v}} + \Psi^{\mathsf{v}}$ and $(\Phi \cap \Psi)^{\mathsf{v}} = \Phi^{\mathsf{v}} \cap \Psi^{\mathsf{v}}$, so that

$$\dim (\Phi +_{\mathsf{A}} \Psi) + \dim (\Phi \cap \Psi) = \dim \Phi + \dim \Psi.$$

On the other hand, supposing that $\Phi \cap \Psi = \emptyset$, we have

$$\dim \left(\Phi +_{\mathsf{A}} \Psi\right) = \dim \Phi + \dim \Psi - \dim \left(\Phi^{\mathsf{v}} \cap \Psi^{\mathsf{v}}\right) + 1.$$

(Assigning \emptyset the standard dimension of -1, the formula above may be written

$$\dim \left(\Phi +_{\mathsf{A}} \Psi\right) + \dim \left(\Phi \cap \Psi\right) = \dim \Phi + \dim \Psi - \dim \left(\Phi^{\mathsf{v}} \cap \Psi^{\mathsf{v}}\right).$$

Hence, no matter whether Φ and Ψ intersect or not, we have

$$\dim (\Phi +_{\mathsf{A}} \Psi) + \dim (\Phi \cap \Psi) \leqslant \dim \Phi + \dim \Psi$$

which is a characterizing inequality for upper semi-modularity.) Note also that we have $\dim (\Phi +_{\mathsf{A}} \Psi) - \dim (\Phi^{\mathsf{v}} + \Psi^{\mathsf{v}}) = 0$ or 1 according as Φ and Ψ intersect or not.

6.6 Affine Independence and Affine Frames

Using an affine structure on a vector space allows any vector to serve as origin. The idea of a dependency in a set of vectors needs to be extended in this light. Here the empty set plays no significant rôle, and it is tacitly assumed henceforth that only nonempty sets and nonempty subsets are being addressed relative to these considerations. A **directional combination** of a finite (nonempty) set of vectors is any linear combination of the set such that the coefficients sum to 0. An **affine dependency** is said to exist in a set of vectors if the zero vector is a nontrivial (scalar coefficients not all zero) directional combination of one of its finite subsets (see also Exercise 6.4). A set in which an affine dependency exists is **affine dependent** and otherwise is **affine independent**. A point is affine independent.

Exercise 6.7 If \mathcal{X} is a nonempty set of vectors and $w \in \mathcal{X}$, then \mathcal{X} is **independent relative to** w if $\{v - w \mid v \in \mathcal{X}, v \neq w\}$ is an independent set of vectors (in the original sense). A nonempty set of vectors is affine independent if and only if it is independent relative to an arbitrarily selected one of its elements.

Exercise 6.8 In an n-dimensional vector space, the number m of elements in an affine independent subset must satisfy $1 \le m \le n+1$ and for each such value of m there exists an affine independent subset having that value of m as its number of elements.

An **affine frame** for the affine flat Φ is an affine independent subset that affine-spans the flat.

Exercise 6.9 An affine frame for the affine flat Φ is a subset of the form $\{v\} \cup (v + \mathcal{B})$ where \mathcal{B} is a basis for the directional subspace Φ^{v} .

Exercise 6.10 Let \mathcal{A} be an affine frame for the affine flat Φ . Then each vector of Φ has a unique expression as an affine combination $\sum_{x \in \mathcal{X}} a_x \cdot x$ where \mathcal{X} is some finite subset of \mathcal{A} and all of the scalars a_x are nonzero.

Exercise 6.11 Let the affine flat Φ have the finite affine frame \mathcal{X} . Then each vector of Φ has a unique expression as an affine combination $\sum_{x \in \mathcal{X}} a_x \cdot x$. The scalars a_x are the barycentric coordinates of the vector relative to the frame \mathcal{X} .

6.7 Affine Maps

Let \mathcal{V} and \mathcal{W} be vector spaces over the same field, which we will always understand to be the case whenever there are to be maps between their affine flats. For $\Phi \in A(\mathcal{V})$ and $\Psi \in A(\mathcal{W})$, we call the function $\alpha : \Phi \to \Psi$ an **affine map** if it preserves translates:

$$\alpha(v+a\cdot(x-w))=\alpha(v)+a\cdot(\alpha(x)-\alpha(w))$$

for all $v, w, x \in \Phi$ and all scalars a.

In a series of exercises we now point out a number of important results concerning affine maps. We lead off with two basic results.

Exercise 6.12 The composite of affine maps is an affine map.

Exercise 6.13 A vector space map is an affine map.

If Φ' , Φ are elements of the same affine structure and $\Phi' \subset \Phi$, we say that Φ' is a **subflat** of Φ . Thus two elements of the same affine structure are incident whenever one is a subflat of the other. The image of an affine map should be an affine flat, if we are even to begin to say that affine structure is preserved. This is true, as the following exercise points out, and as a consequence subflats are also sent to subflats with the result that incidence is completely preserved.

Exercise 6.14 The image of an affine flat under an affine map is an affine flat. Since the restriction of an affine map to a subflat is clearly an affine map, an affine map therefore maps subflats to subflats and thereby preserves incidence.

Affine expressions are preserved as well, and the preservation of affine spans and affine sums then follows.

Exercise 6.15 An affine map preserves all affine expressions of vectors of its domain. Hence under an affine map, the image of the affine span of a subset of the domain is the affine span of its image, and affine maps preserve affine sums.

A one-to-one function clearly preserves all intersections. However, in general it can only be said that the intersection of subset images contains the image of the intersection of the subsets. Thus it should not be too suprising that affine maps do not necessarily even preserve the intersection of flats as the following exercise demonstrates.

Exercise 6.16 As usual, let \mathcal{F} be a field. Let $\alpha: \mathcal{F}^2 \to \mathcal{F}^2$ be the vector space map that sends (a,b) to (0,a+b). Let $\Phi = \{(a,0) \mid a \in \mathcal{F}\}$ and $\Psi = \{(0,a) \mid a \in \mathcal{F}\}$. Then $\alpha(\Phi \cap \Psi) \neq \alpha(\Phi) \cap \alpha(\Psi)$.

A basic result about vector space maps generalizes to affine maps.

Exercise 6.17 The inverse of a bijective affine map is an affine map.

The bijective affine maps are therefore isomorphisms, so that $\Phi \in A(\mathcal{V})$ and $\Psi \in A(\mathcal{W})$ are isomorphic (as affine flats) when there is a bijective affine map from one to the other. Isomorphic objects, such as isomorphic affine flats, are always **aliases** of each other, with context serving to clarify the type of object.

Affine maps and vector space maps are closely related, as the following theorem details.

Theorem 75 Let α be an affine map from $\Phi \in A(\mathcal{V})$ to $\Psi \in A(\mathcal{W})$ and fix an element $v \in \Phi$. Let $f : \Phi^{\mathsf{v}} \to \Psi^{\mathsf{v}}$ be the function defined by

$$f(h) = \alpha (v + h) - \alpha (v)$$

for all $h \in \Phi^{\mathsf{v}}$. Then f is a vector space map that is independent of the choice of v.

Proof: Let $x, w \in \Phi$ be arbitrary, so that h = x - w is an arbitrary element of Φ^{v} . Then

$$f(a \cdot h) = \alpha (v + a \cdot (x - w)) - \alpha (v) = a \cdot \alpha (x) - a \cdot \alpha (w) =$$

$$= a \cdot (\alpha (v) + \alpha (x) - \alpha (w)) - a \cdot \alpha (v) =$$

$$= a \cdot (\alpha (v + h) - \alpha (v)) = a \cdot f(h).$$

Similarly, we find that f(h+k) = f(h) + f(k) for arbitrary $h, k \in \Phi^{\mathsf{v}}$. Hence f is a vector space map. Replacing h with x-w, we find that $f(h) = \alpha(v+x-w) - \alpha(v) = \alpha(x) - \alpha(w)$, showing that f(h) is actually independent of v.

The vector space map f that corresponds to the affine map α as in the theorem above is the **underlying vector space map** of α , which we will denote by α^{v} . Thus for any affine map α we have

$$\alpha\left(v\right) = \alpha^{\mathsf{v}}\left(v-u\right) + \alpha\left(u\right)$$

for all u, v in the domain of α .

Exercise 6.18 Let $\Phi \in A(V)$ and $\Psi \in A(W)$ and let $f : \Phi^{\mathsf{v}} \to \Psi^{\mathsf{v}}$ be a given vector space map. Fix an element $v \in \Phi$ and let $\alpha : \Phi \to \Psi$ be the function such that for each $h \in \Phi^{\mathsf{v}}$, $\alpha(v+h) = \alpha(v) + f(h)$. Then α is an affine map such that $\alpha^{\mathsf{v}} = f$.

Other nice properties of affine maps are now quite apparent, as the following exercises detail.

Exercise 6.19 Under an affine map, the images of strictly parallel affine flats are strictly parallel affine flats.

Exercise 6.20 An affine map is completely determined by its values on any affine spanning set, and if that affine spanning set is an affine frame, the values may be arbitrarily assigned.

Exercise 6.21 Let $\alpha : \Phi \to \Psi$ be an affine map from $\Phi \in A(\mathcal{V})$ to $\Psi \in A(\mathcal{W})$. Then there exists the affine map $\alpha^{\sharp} : \mathcal{V} \to \Psi$ that agrees with α on Φ .

6.8 Some Affine Self-Maps

An affine self-map $\delta_{t,u;b}:\Phi\to\Phi$ of the flat Φ that sends $v\in\Phi$ to

$$\delta_{t,u;b}(v) = t + b \cdot (v - u)$$
,

where t, u are fixed in Φ and b is a fixed scalar, is known as a **dilation** (or **dilatation**, a term we will not use, but which means the same). Every **proper** dilation ($b \neq 0$) is invertible: $\delta_{t,u;b}^{-1} = \delta_{u,t;b^{-1}}$. The following result tells us that the proper dilations are the direction-preserving automorphisms of affine flats.

Proposition 76 A function $\delta : \Phi \to \Phi$ on the affine flat Φ is a dilation $\delta_{t,u;b}$ if and only if $\delta(v) - \delta(w) = b \cdot (v - w)$ for all $v, w \in \Phi$.

Proof: A dilation clearly satisfies the condition. On the other hand, suppose that the condition holds. Fix $u, w \in \Phi$ and define $t \in \Phi$ by $t = \delta(w) - b \cdot (w - u)$. Then, employing the condition, we find that for any $v \in \Phi$, $\delta(v) = \delta(w) + b \cdot (v - w) = t + b \cdot (w - u) + b \cdot (v - w) = t + b \cdot (v - u)$.

Letting h = t - u and

$$\tau_h(v) = v + h = \delta_{t,u:1}(v)$$

for all $v \in \Phi$ gives the special dilation $\tau_h : \Phi \to \Phi$ which we call a **translation** by h. Every translation is invertible, $\tau_h^{-1} = \tau_{-h}$, and τ_0 is the identity map. The translations on Φ correspond one-to-one with the vectors of the directional subspace Φ^{v} of which Φ is a coset. We have $\tau_h \circ \tau_k = \tau_k \circ \tau_h = \tau_{k+h}$ and thus under the operation of composition, the translations on Φ are isomorphic to the additive group of Φ^{v} . Since every translation is a dilation $\delta_{t,u;b}$ with b=1, from the proposition above we infer the following result.

Corollary 77 A function $\tau : \Phi \to \Phi$ on the affine flat Φ is a translation if and only if $\tau(v) - \tau(w) = v - w$ for all $v, w \in \Phi$.

Exercise 6.22 The translations are the dilations that lack a unique fixed point (or "center").

6.9 Congruence Under the Affine Group

Viewing the vector space \mathcal{V} as an affine flat in $A(\mathcal{V})$ and using the composition of maps as group operation, the bijective affine self-maps $\alpha: \mathcal{V} \to \mathcal{V}$ form a group $\mathsf{GA}(\mathcal{V})$ called the **affine group** of \mathcal{V} . The elements of $\mathsf{GA}(\mathcal{V})$ may be described as the invertible vector space self-maps on \mathcal{V} composed with the translations on \mathcal{V} . Similarly, there is an affine group $\mathsf{GA}(\Phi)$ for any affine flat $\Phi \in \mathsf{A}(\mathcal{V})$. However, $\mathsf{GA}(\mathcal{V})$ already covers the groups $\mathsf{GA}(\Phi)$ in the sense that any such $\mathsf{GA}(\Phi)$ may be viewed as the restriction to Φ of the subgroup of $\mathsf{GA}(\mathcal{V})$ that fixes Φ (those maps for which the image of Φ remains contained in Φ).

A figure in \mathcal{V} is just geometric language for a subset of \mathcal{V} . Two figures in \mathcal{V} are called **affine congruent** if one is the image of the other under an element of $\mathsf{GA}(\mathcal{V})$. Affine congruence is an equivalence relation.

Exercise 6.23 Affine frames in a vector space are affine congruent if and only if they have the same cardinality.

Exercise 6.24 Affine flats in a vector space are affine congruent if and only if their directional subspaces have bases of the same cardinality.

6.10 Problems

1. When \mathcal{F} is the smallest possible field, namely the field $\{0,1\}$ of just two elements, any nonempty subset \mathcal{S} of a vector space \mathcal{V} over \mathcal{F} contains the translate $v + a \cdot (x - v)$ for every $v, x \in \mathcal{S}$ and every $a \in \mathcal{F}$. Not all such \mathcal{S} are affine flats, however.

On the other hand, if $1+1 \neq 0$ in \mathcal{F} , a nonempty subset \mathcal{S} of a vector space over \mathcal{F} is an affine flat if it contains $v+a\cdot(x-v)$ for every $v,x\in\mathcal{S}$ and every $a\in\mathcal{F}$.

What about when \mathcal{F} is a field of 4 elements?

7 Basic Affine Results and Methods

7.1 Notations

Throughout this chapter, \mathcal{V} is our usual vector space over the field \mathcal{F} . We generally omit any extra assumptions regarding \mathcal{V} or \mathcal{F} (such as dim $\mathcal{V} \neq 0$) that each result might need to make its hypotheses realizable. Without further special mention, we will frequently use the convenient abuse of notation $P = \{P\}$ so that the point $P \in \mathcal{A}(\mathcal{V})$ is the singleton set that contains the vector $P \in \mathcal{V}$.

7.2 Two Axiomatic Propositions

The two propositions presented in this section are of an extremely basic geometric character, and are assumed as axioms in most developments of affine geometry from the *synthetic* viewpoint. While certainly not universal, the first of these does apply to many ordinary kinds of geometries. It is commonly, but somewhat loosely, expressed as "two points determine a line."

Proposition 78 There is one and only one line that contains the distinct points P and Q.

Proof: The affine flat $k = \{P + a \cdot (Q - P) \mid a \in \mathcal{F}\}$ clearly contains both P and Q, and is a line since $k^{\mathsf{v}} = \{a \cdot (Q - P) \mid a \in \mathcal{F}\}$ is one-dimensional. On the other hand, suppose that l is a line that contains both P and Q. Since l is a line that contains P, it has the form $l = P + l^{\mathsf{v}}$ where l^{v} is one-dimensional. Since l is an affine flat that contains both P and Q, it contains the translate P + (Q - P) so that $Q - P \in l^{\mathsf{v}}$, and therefore $l^{\mathsf{v}} = k^{\mathsf{v}}$ and l = k.

The line that contains the distinct points P and Q is $P +_{\mathsf{A}} Q$, of course. We will write PQ to mean the line $P +_{\mathsf{A}} Q$ for (necessarily distinct) points P and Q.

Corollary 79 Two distinct intersecting lines intersect in a single point.

Exercise 7.1 Dimensional considerations then imply that the affine sum of distinct intersecting lines is a plane.

Our second proposition involves parallelism, and can be used in a synthetic context to prove the transitivity of parallelism for coplanar lines.

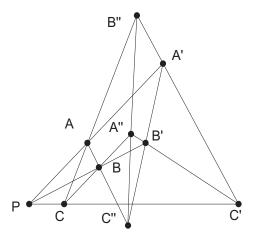
Proposition 80 Given a point P and a line l, there is one and only one line m that contains P and is parallel to l.

Proof: In order for m and l to be parallel, we must have $m^{\mathsf{v}} = l^{\mathsf{v}}$, since m and l have equal finite dimension. Hence the unique line sought is $m = P + l^{\mathsf{v}}$. (Recalling that the cosets of a subspace form a partition of the full space, P must lie in exactly one coset of l^{v} .)

7.3 A Basic Configuration Theorem

The following is one of the affine versions of a configuration theorem of projective geometry attributed to the French geometer Girard Desargues (1591-1661). The particular affine version treated here is the completely nonparallel one, the one that looks just like the projective theorem as we shall later see.

Theorem 81 (Desargues) Let A, A', B, B', C, C' be distinct points and let the distinct lines AA', BB', and CC' be concurrent in the point P. Let the nonparallel lines AB and A'B' intersect in the point C'', the nonparallel lines BC and B'C' intersect in the point A'', and the nonparallel lines CA and C'A' intersect in the point B''. Then A'', B'', and C'' are collinear.



Illustrating the Theorem of Desargues

Proof: We have $P = a \cdot A + a' \cdot A' = b \cdot B + b' \cdot B' = c \cdot C + c' \cdot C'$ for scalars a, a', b, b'c, c' which satisfy a + a' = b + b' = c + c' = 1. We then

derive the three equalities

$$a \cdot A - b \cdot B = b' \cdot B' - a' \cdot A',$$

$$b \cdot B - c \cdot C = c' \cdot C' - b' \cdot B',$$

$$c \cdot C - a \cdot A = a' \cdot A' - c' \cdot C'.$$

Consider the first of these. Suppose that a=b. Then a'=b' too, since a+a'=b+b'=1. But then $a\neq 0$, for if a=0, then a'=1, which would give A'=B' contrary to hypothesis. Similarly $a'\neq 0$. Thus supposing that a=b leads to the nonparallel lines AB and A'B' being parallel. Hence $a-b=b'-a'\neq 0$, and similarly, $b-c=c'-b'\neq 0$ and $c-a=a'-c'\neq 0$. Denote these pairs of equal nonzero differences by c'', a'', and b'', respectively. Then the equalities above relate to the double-primed intersection points by

$$a \cdot A - b \cdot B = b' \cdot B' - a' \cdot A' = c'' \cdot C'',$$

 $b \cdot B - c \cdot C = c' \cdot C' - b' \cdot B' = a'' \cdot A'',$
 $c \cdot C - a \cdot A = a' \cdot A' - c' \cdot C' = b'' \cdot B'',$

since if we divide by the double-primed scalar, each double-primed intersection point is the equal value of two affine expressions that determine points on the intersecting lines. Adding up the equalities gives $0 = a'' \cdot A'' + b'' \cdot B'' + c'' \cdot C''$ and noting that a'' + b'' + c'' = 0 we conclude in light of Exercise 6.4 that A'', B'', and C'' do lie on one line as was to be shown.

7.4 Barycenters

We now introduce a concept analogous to the physical concept of "center of gravity," which will help us visualize the result of computing a linear expression and give us some convenient notation. Given the points A_1, \ldots, A_n , consider the linear expression $a_1 \cdot A_1 + \cdots + a_n \cdot A_n$ of weight $a_1 + \cdots + a_n \neq 0$ based on A_1, \ldots, A_n . The **barycenter** based on (the factors of the terms of) this linear expression is the unique point X such that

$$(a_1 + \dots + a_n) \cdot X = a_1 \cdot A_1 + \dots + a_n \cdot A_n.$$

(If the weight were zero, there would be no such unique point X.) The barycenter of $a_1 \cdot A_1 + \cdots + a_n \cdot A_n$, which of course lies in the affine span of $\{A_1, \ldots, A_n\}$, will be denoted by $\mathcal{L}[a_1 \cdot A_1 + \cdots + a_n \cdot A_n]$. Clearly, for any

scalar $m \neq 0$, $\mathcal{L}[(ma_1) \cdot A_1 + \cdots + (ma_n) \cdot A_n] = \mathcal{L}[a_1 \cdot A_1 + \cdots + a_n \cdot A_n]$, and this homogeneity property allows us to use $\mathcal{L}[a_1 \cdot A_1 + \cdots + a_n \cdot A_n]$ as a convenient way to refer to an affine expression by referring to any one of its nonzero multiples instead. Also, supposing that $1 \leq k < n$, $a = a_1 + \cdots + a_k \neq 0$, $a' = a_{k+1} + \cdots + a_n \neq 0$,

$$X_k = \pounds \left[a_1 \cdot A_1 + \dots + a_k \cdot A_k \right],$$

and

$$X_k' = \mathcal{L}\left[a_{k+1} \cdot A_{k+1} + \dots + a_n \cdot A_n\right],$$

it is easy to see that if $a + a' \neq 0$ then

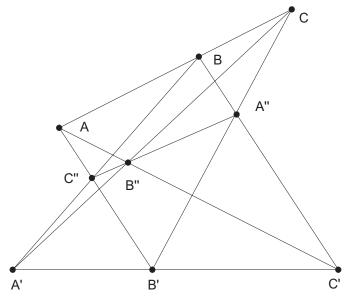
$$\mathcal{L}\left[a_1 \cdot A_1 + \dots + a_n \cdot A_n\right] = \mathcal{L}\left[a \cdot X_k + a' \cdot X_k'\right].$$

Hence we may compute barycenters in a piecemeal fashion, provided that no zero weight is encountered along the way.

7.5 A Second Basic Configuration Theorem

The following is the affine version of a configuration theorem of projective geometry which dates back much farther than the theorem of Desargues given above. Pappus of Alexandria, a Greek who lived in the 4th century, long before projective geometry was created, used concepts of Euclidean geometry to prove this same affine version of the projective theorem that now bears his name.

Theorem 82 (Theorem of Pappus) Let l, l' be distinct coplanar lines with distinct points $A, B, C \subset l \setminus l'$ and distinct points $A', B', C' \subset l' \setminus l$. If BC' meets B'C in A'', AC' meets A'C in B'', and AB' meets A'B in C'', then A'', B'', C'' are collinear.



Illustrating the Theorem of Pappus

Proof: We will express A'', B'', C'' in terms of the elements of the affine frame $\{A, B', C\}$. First, we may write for some a, b', c with a + b' + c = 1,

$$B'' = \pounds [a \cdot A + b' \cdot B' + c \cdot C].$$

Because B is on AC, we may write

$$B = \pounds \left[pa \cdot A + c \cdot C \right].$$

Because A' is on B''C, we may write

$$A' = \mathcal{L} [B'' + (q-1)c \cdot C]$$

= $\mathcal{L} [a \cdot A + b' \cdot B' + qc \cdot C]$.

Because C' is on both AB'' and A'B', we may write $C'=\pounds\left[r\cdot A+q\cdot B''\right]=\pounds\left[A'+s\cdot B'\right]$ or

$$C' = \mathcal{L} [(r+qa) \cdot A + qb' \cdot B' + qc \cdot C]$$

=
$$\mathcal{L} [a \cdot A + (b'+s) \cdot B' + qc \cdot C]$$

and selecting between equal coefficients permits us to write

$$C' = \pounds [a \cdot A + qb' \cdot B' + qc \cdot C].$$

Because A'' is on both BC' and B'C, we may write $A'' = \pounds [p \cdot C' - k \cdot B] = \pounds [w \cdot B' + x \cdot C]$ or

$$A'' = \mathcal{L} [(1-k)pa \cdot A + pqb' \cdot B' + (pq-1)c \cdot C]$$

= $\mathcal{L} [w \cdot B' + x \cdot C]$

so that k = 1 and

$$A'' = \pounds \left[pqb' \cdot B' + (pq - 1)c \cdot C \right].$$

Because C'' is on both A'B and AB', we may write $C'' = \pounds [l \cdot A' - q \cdot B] = \pounds [y \cdot A + z \cdot B']$ or

$$C'' = \mathcal{L} [(l - qp)a \cdot A + lb' \cdot B' + (l - 1)qc \cdot C]$$

= $\mathcal{L} [y \cdot A + z \cdot B']$

so that l = 1 and

$$C'' = \pounds \left[(1 - qp)a \cdot A + b' \cdot B' \right].$$

Letting a'', b'', c'' be the respective weights of the expressions above of which A'', B'', C'' are barycenters in terms of A, B', C, we have

$$a'' = pqb' + (pq - 1)c,$$

 $b'' = a + b' + c,$
 $c'' = (1 - qp)a + b'.$

We then find that

$$a'' \cdot A'' + (1 - pq)b'' \cdot B'' + (-c'') \cdot C'' = 0$$

and the left-hand side is a directional expression. However, the coefficients of this directional expression are not all zero. For if we suppose they are all zero, then 1 - pq = 0, and b' = 0, which puts B'' on AC so that C' would be on AC contrary to hypothesis. We conclude in light of Exercise 6.4 that A'', B'', and C'' are collinear as was to be shown.

7.6 Vector Ratios

If two nonzero vectors u, v satisfy $a \cdot u = b \cdot v$ for two nonzero scalars a, b, it makes sense to speak of the proportion u : v = b : a. Under these conditions we say that u, v have the **same direction** and only then do we impute a scalar value to the **vector ratio** $\frac{u}{v}$ and write $\frac{u}{v} = \frac{b}{a}$.

Let us write $\overrightarrow{PQ} = Q - P$ whenever P and Q are points. Nonzero vectors \overrightarrow{PQ} and \overrightarrow{RS} have the same direction precisely when PQ and RS are parallel lines. Hence the ratio $\overrightarrow{PQ}/\overrightarrow{RS}$ has meaning whenever PQ and RS are parallel lines, and in particular, when they are the same line.

X is a point on the line AB, and distinct from A and B, if and only if there are nonzero scalars a, b, such that $a+b \neq 0$ and such that $X = \pounds[a \cdot A + b \cdot B]$. But this is the same as saying that there are nonzero scalars a and b with nonzero sum, such that $(a+b) \cdot \overrightarrow{AX} = b \cdot \overrightarrow{AB}$ and $(a+b) \cdot \overrightarrow{XB} = a \cdot \overrightarrow{AB}$, or equivalently such that

$$\frac{\overrightarrow{AX}}{\overrightarrow{XB}} = \frac{b}{a}.$$

We say that $\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$ is **divided** by X in the ratio b: a.

The result of the following Subcenter Exercise will find useful employment below as a lemma.

Exercise 7.2 (Subcenter Exercise) Given noncollinear points A, B, C and nonzero scalars a, b, c such that $X = \pounds [a \cdot A + b \cdot B + c \cdot C]$, barycentric considerations imply that when $a + b \neq 0$, CX and AB meet in $\pounds [a \cdot A + b \cdot B]$. On the other hand, when a + b = 0, CX and AB are parallel. Hence $CX \parallel AB$ if and only if a + b = 0, and $CX \cap AB = \pounds [a \cdot A + b \cdot B]$ if and only if $a + b \neq 0$.

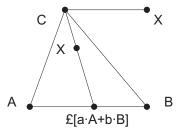


Figure for Above Exercise

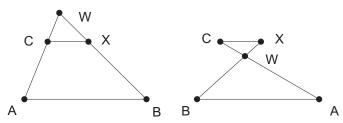
The next result is the affine version of Proposition VI.2 from Euclid's *Elements*. By PQR we will always mean the plane that is the affine sum of three given (necessarily noncollinear) points P, Q, R.

Theorem 83 (Similarity Theorem) Let X be a point in ABC and not on AB, BC, or CA, and let $W = BX \cap AC$. Then

$$CX \parallel AB \Rightarrow \frac{\overrightarrow{WC}}{\overrightarrow{WA}} = \frac{\overrightarrow{WX}}{\overrightarrow{WB}} = \frac{\overrightarrow{CX}}{\overrightarrow{AB}}$$

and

$$\frac{\overrightarrow{WC}}{\overrightarrow{WA}} = \frac{\overrightarrow{WX}}{\overrightarrow{WB}} \Rightarrow CX \parallel AB.$$



Two Possible Figures for the Similarity Theorem

Proof: "Let X be a point in ABC and not on AB, BC, or CA" is just the geometric way to say that there exist nonzero scalars a,b,c such that $X = \pounds [a \cdot A + b \cdot B + c \cdot C]$. Suppose first that $CX \parallel AB$. Then by the Subcenter Exercise above, a + b = 0 and we may replace a with -b to give $X = \pounds [(-b) \cdot A + b \cdot B + c \cdot C]$ or $c \cdot (X - C) = b \cdot (B - A)$. Hence $\overrightarrow{CX}/\overrightarrow{AB} = b/c$. Employing the Subcenter Exercise again, we find that $W = BX \cap AC = \pounds [(-b) \cdot A + c \cdot C]$ so that W divides \overrightarrow{AC} in the ratio c : -b and therefore $\overrightarrow{WC}/\overrightarrow{WA} = b/c$. X may also be obtained by the piecemeal calculation $X = \pounds [(c - b) \cdot W + b \cdot B]$ which is the same as $c \cdot (X - W) = b \cdot (B - W)$ or $\overrightarrow{WX}/\overrightarrow{WB} = b/c$. This proves the first part.

On the other hand, assume that $\overrightarrow{WC}/\overrightarrow{WA} = \overrightarrow{WX}/\overrightarrow{WB}$. We may suppose that these two equal vector ratios both equal b/c for two nonzero scalars b, c. Then $c \cdot (C - W) = b \cdot (A - W)$ and $c \cdot (X - W) = b \cdot (B - W)$. Subtracting the first of these two equations from the second, we obtain $c \cdot (X - C) = b \cdot (B - A)$, which shows that $CX \parallel AB$ as desired.

7.7 A Vector Ratio Product and Collinearity

The following theorem provides a vector ratio product criterion for three points to be collinear. The Euclidean version is attributed to Menelaus of Alexandria, a Greek geometer of the late 1st century.

Theorem 84 (Theorem of Menelaus) Let A, B, C be noncollinear points, and let A', B', C' be points none of which is A, B, or C, such that C' is on AB, A' is on BC, and B' is on CA. Then A', B', C' are collinear if and only if

$$\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}} \cdot \frac{\overrightarrow{CB'}}{\overrightarrow{B'A}} \cdot \frac{\overrightarrow{AC'}}{\overrightarrow{C'B}} = -1.$$

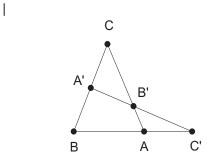


Figure for the Theorem of Menelaus

Proof: Let A', B', C' be collinear. B, C, C' are noncollinear and we may write $B' = \pounds [b \cdot B + c \cdot C + c' \cdot C']$. Then $A' = \pounds [b \cdot B + c \cdot C]$ and $A = \pounds [b \cdot B + c' \cdot C']$ by the Subcenter Exercise above. B' may also be obtained by the piecemeal calculation $B' = \pounds [c \cdot C + (b + c') \cdot A]$. Thus A' divides \overrightarrow{BC} in the ratio c: b, B' divides \overrightarrow{CA} in the ratio (b + c'): c, and A divides $\overrightarrow{BC'}$ in the ratio c': b. Hence

$$\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}} \cdot \frac{\overrightarrow{CB'}}{\overrightarrow{B'A}} \cdot \frac{\overrightarrow{AC'}}{\overrightarrow{C'B}} = \frac{c}{b} \cdot \frac{b+c'}{c} \cdot \frac{b}{-b-c'} = -1$$

as required.

On the other hand, suppose that the product of the ratios is -1. We may suppose that

$$A' = \pounds \left[1 \cdot B + x \cdot C \right], B' = \pounds \left[y \cdot A + x \cdot C \right], C' = \pounds \left[y \cdot A + z \cdot B \right]$$

where none of x, y, z, 1 + x, y + x, y + z is zero. Thus

$$\frac{x}{1} \cdot \frac{y}{x} \cdot \frac{z}{y} = -1$$

and hence z = -1. Then we have

$$(1+x) \cdot A' + (-y-x) \cdot B' + (y-1) \cdot C' = 0$$

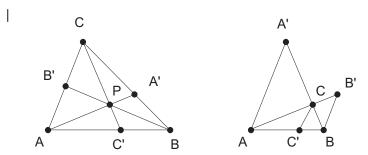
and (1+x)+(-y-x)+(y-1)=0. We conclude in light of Exercise 6.4 that A', B', C' are collinear as required.

7.8 A Vector Ratio Product and Concurrency

The Italian geometer Giovanni Ceva (1647-1734) is credited with the Euclidean version of the following theorem which is quite similar to the above theorem of Menelaus but instead provides a criterion for three lines to be either concurrent or parallel.

Theorem 85 (Theorem of Ceva) Let A, B, C be noncollinear points, and let A', B', C' be points none of which is A, B, or C, such that C' is on AB, A' is on BC, and B' is on CA. Then AA', BB', CC' are either concurrent in a point not lying on AB, BC, or CA, or are parallel, if and only if

$$\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}} \cdot \frac{\overrightarrow{CB'}}{\overrightarrow{B'A}} \cdot \frac{\overrightarrow{AC'}}{\overrightarrow{C'B}} = 1.$$



Figures for the Theorem of Ceva

Proof: Suppose AA', BB', CC' meet in $P = \pounds [a \cdot A + b \cdot B + c \cdot C]$ with nonzero scalars a, b, c. By the Subcenter Exercise above, $A' = \pounds [b \cdot B + c \cdot C]$

so that $\overrightarrow{BA'}/\overrightarrow{A'C} = c/b$. Similarly, we find that $\overrightarrow{CB'}/\overrightarrow{B'A} = a/c$ and $\overrightarrow{AC'}/\overrightarrow{C'B} = b/a$. The desired product result follows at once.

Suppose next that AA', BB', CC' are parallel. Applying the Similarity Theorem above, $AA' \parallel CC'$ implies that C divides A'B in the same ratio as C' divides AB, and $BB' \parallel CC'$ implies that C also divides AB' in the same ratio as C' divides AB. The desired product result follows readily.

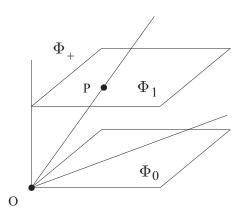
Now suppose that the product is as stated. According to the conditions under which we allow vector ratios to be written, there are nonzero scalars a,b,c such that $\overrightarrow{BA'}/\overrightarrow{A'C} = c/b$, $\overrightarrow{CB'}/\overrightarrow{B'A} = a/c$, and $\overrightarrow{AC'}/\overrightarrow{C'B} = b/a$. If $a+b+c\neq 0$, define $P=\pounds\left[a\cdot A+b\cdot B+c\cdot C\right]$. Then \overrightarrow{BC} is divided by A' in the ratio $c:b,A'=\pounds\left[b\cdot B+c\cdot C\right]$, and barycentric considerations place P on AA'. Similarly, P is on BB' and CC'. If, on the other hand, a+b+c=0, then c=-a-b. We see then that C divides A'B in the same ratio as C' divides AB, and that C also divides AB' in the same ratio as C' divides AB. The Similarity Theorem above then gives $AA' \parallel CC'$ and $BB' \parallel CC'$.

8 An Enhanced Affine Environment

8.1 Inflating a Flat

Let \mathcal{V} be our usual vector space over the field \mathcal{F} . Any flat $\Phi \in A(\mathcal{V})$ may be embedded in a vector space Φ_+ as a hyperplane not containing 0, a construction that we will call **inflating** Φ . Employing the directional subspace Φ^{v} of Φ , we form $\Phi_{+} = \mathcal{F} \times \Phi^{\mathsf{v}}$ and embed Φ in it as $\{1\} \times \Phi^{\mathsf{v}}$ by the affine isomorphism that first sends $x \in \Phi$ to $x^{\mathsf{v}} = (x - o) \in \Phi^{\mathsf{v}}$ and then sends x^{v} to $1 \times x^{\mathsf{v}} = (1, x^{\mathsf{v}})$, where o is a fixed but arbitrary vector in Φ . Forming Φ_{+} begins a transition from affine to projective geometry. Benefits of this enhancement of the affine environment include more freedom in representing points (similar to that gained by using the \mathcal{L} operator) and the ability to completely separate the vectors representing points from the vectors representing directions. Having then gained the separation of point-representing vectors and direction-representing vectors, the directions can be added to the point community as a new special class of generalized points which from the affine viewpoint are thought of as lying at infinity. These points at infinity can be used to deal in a uniform way with cases where an intersection point of lines disappears to infinity as the lines become parallel. This leads us into the viewpoint of projective geometry where what were points at infinity become just ordinary points, and parallel lines cease to exist.

We now assume that any flat Φ which is of geometric interest to us has been inflated, so that the rôle of Φ will be played by its alias $\{1\} \times \Phi^{\mathsf{v}}$ in $A(\Phi_+)$, and the rôle of Φ^{\vee} then will be played by its alias $\{0\} \times \Phi^{\vee}$. To simplify the notation, we will refer to $\{a\} \times \Phi^{\mathsf{v}}$ as Φ_a , so that Φ_1 is playing the rôle of Φ and Φ_0 is playing the rôle of Φ^{v} . The rôle of any subflat $\Psi \subset \Phi$ is played by the subflat $\Psi_1 = \{1\} \times \Psi^{\mathsf{v}} \subset \Phi_1$, of course. No matter what holds for Φ and its directional subspace, Φ_1 is disjoint from its directional subspace Φ_0 . Lines through the point $O = \{0\}$ in Φ_+ are of two fundamentally different types with regard to Φ_1 : either such a line meets Φ_1 or it does not. The nonzero vectors of the lines through O that meet Φ_1 will be called **point vectors**, and the nonzero vectors of the remaining lines through O (those that lie in Φ_0) will be called **direction vectors**. There is a unique line through O passing through any given point P of Φ_1 , and since any nonzero vector in that line may be used as an identifier of P, that line and each of its point vectors will be said to **represent** P. Point vectors are homogeneous representatives of their points in the sense that multiplication by any nonzero scalar gives a result that still represents exactly the same point. To complete our view, each line through O lying in Φ_0 will be called a **direction**, and it and each of its nonzero vectors will be said to **represent** that direction. Thus the zero vector stands alone in not representing either a point or a direction.



Illustrating Inflating Φ

We are now close to the viewpoint of projective geometry where the lines through 0 actually are the points, and the n-dimensional subspaces are the (n-1)-dimensional flats of a projective structure. However, the inflated-flat viewpoint is still basically an affine viewpoint, but an enhanced one that is close to the pure projective one. From now on, we assume that any affine flat has been inflated and freely use the resulting enhanced affine viewpoint.

Suppose now that $P, Q, R \in \Phi_1$ are distinct points represented by the point vectors p, q, r. Our geometric intuition indicates that P, Q, R are collinear in Φ_1 if and only if p, q, r are coplanar in Φ_+ . The following proposition confirms this.

Proposition 86 Let $P, Q, R \in \Phi_1$ be distinct points represented by the point vectors p, q, r. Then P, Q, R are collinear if and only if $\{p, q, r\}$ is dependent.

Proof: Let i denote the vector 1×0 in $\Phi_+ = \mathcal{F} \times \Phi^{\mathsf{v}}$. Then points in Φ_1 all have the form $\{i+x\}$ where $x \in \Phi_0$, and point vectors all have the form $d \cdot (i+x)$ for some nonzero scalar d. We thus set $P = \{i+u\}, Q = \{i+v\}, R = \{i+w\}$, and $p = a \cdot (i+u), q = b \cdot (i+v), r = c \cdot (i+w)$. Supposing that P, Q, R are collinear, there are nonzero scalars k, l, m such that $k \cdot (i+u) + l \cdot (i+v) + m \cdot (i+w) = 0$. Therefore p, q, r are related by

$$\frac{k}{a} \cdot p + \frac{l}{b} \cdot q + \frac{m}{c} \cdot r = 0.$$

On the other hand, suppose that there are scalars f, g, h, not all zero, such that $f \cdot p + g \cdot q + h \cdot r = 0$. Then

$$fa \cdot (i+u) + gb \cdot (i+v) + hc \cdot (i+w) = 0.$$

Since $i \notin \Phi_0$, fa + gb + hc = 0 and P, Q, R are collinear by Exercise 6.4.

Note that since we have complete freedom in scaling point vectors by nonzero scalars, if we are given distinct collinear P, Q, R, there always is a $\{p, q, r\}$ such that the scalar coefficients of a dependency linear combination of it are any three nonzero scalars we choose. Moreover, since a dependency linear combination is unaffected by a nonzero scaling, besides the three nonzero scalar coefficients, one of p, q, r may also be picked in advance. This is the freedom in representing points which we get from having insured that

the flat where all the points lie is completely separated from its directional subspace.

If we are given two distinct points P,Q represented by point vectors p,q, the above proposition implies that any point vector in the span of $\{p,q\}$ represents a point on the line PQ, and conversely, that every point of PQ in Φ_1 is represented by a point vector of $\langle \{p,q\} \rangle$. But besides point vectors, there are other nonzero vectors in $\langle \{p,q\} \rangle$, namely direction vectors. It is easy to see that the single direction contained in $\langle \{p,q\} \rangle$ is $V = \langle \{p,q\} \rangle \cap \Phi_0$ which we refer to as the **direction** of PQ. We will consider V to be a generalized point of PQ, and we think of V as lying "at infinity" on PQ. Doing this will allow us to treat all nonzero vectors of $\langle \{p,q\} \rangle$ in a uniform manner and to say that $\langle \{p,q\} \rangle$ represents the line PQ. This will turn out to have the benefit of eliminating the need for separate consideration of related "parallel cases" for many of our results.

Exercise 8.1 Let P,Q be distinct points represented by point vectors p,q, and let the direction V be represented by the direction vector v. Then the lines $\langle \{p,v\} \rangle \cap \Phi_1$ and $\langle \{q,v\} \rangle \cap \Phi_1$ are parallel and $\langle \{p,v\} \rangle \cap \langle \{q,v\} \rangle = \langle \{v\} \rangle$. (Thus we speak of the parallel lines PV and QV that meet at V.)

Besides lines like PV and QV of the above exercise, there can be lines of the form VW where V and W are distinct directions. We speak of such lines as lines at infinity. If V is represented by v and W by w, then the nonzero vectors of $\langle \{v, w\} \rangle$ (all of which are direction vectors) are exactly the vectors that represent the points of VW. Thus we view the "compound direction" $\langle \{v, w\} \rangle$ as a generalized line. Each plane has its line at infinity, and planes have the same line at infinity if and only if they are parallel.

8.2 Desargues Revisited

We now state and prove another version of the theorem of Desargues treated previously. The theorem's points and lines are now assumed to exist in our new generalized framework.

Theorem 87 (Desargues) Let A, A', B, B', C, C', P be distinct points and let the lines AA', BB', and CC' be distinct and concurrent in P. Let AB and A'B' meet in the point C'', BC and B'C' meet in the point A'', and CA and C'A' meet in the point B''. Then A'', B'', and C'' all lie on the same line.

Proof: The corresponding small letter will always stand for a representing vector. We can find representing vectors such that

$$p = a + a' = b + b' = c + c'$$

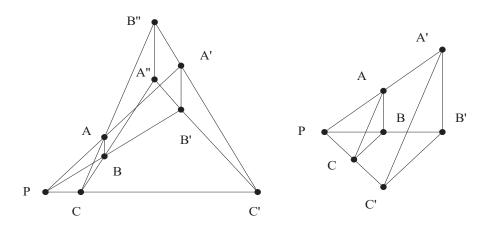
so that

$$a - b = b' - a' = c''$$

 $b - c = c' - b' = a''$
 $c - a = a' - c' = b''$.

Hence a'' + b'' + c'' = 0.

While we were able to use simpler equations, this proof is not essentially different from the previous one. What is more important is that new interpretations now arise from letting various of the generalized points be at infinity. We depict two of these possible new interpretations in the pair of figures below.



New Interpretations Arising from Generalized-Framework Version

In the left-hand figure, notice that A''B'' is parallel to the parallel lines AB and A'B', as it contains their intersection point C'' at infinity. In the right-hand figure, A'', B'', and C'' all lie on the same line at infinity, and we can say that the fact that any two of A'', B'', C'' lie at infinity means that the third must also. Thus the right-hand figure depicts the result that if any two of the three corresponding pairs AB, A'B', etc., are pairs of parallels, then the third is also.

8.3 Representing (Generalized) Subflats

Notice: The qualifier "generalized" will henceforth usually be omitted. From now on, points of Φ or other subflats to which we refer may conceivably be "at infinity" unless specifically stated otherwise. This includes intersections, so that, for example, parallel lines will be deemed to intersect in a point just like nonparallel coplanar lines. Similarly, the more or less obvious generalizations of all encountered affine concepts (affine span, affine sum, etc.) are to be assumed to be in use whether specifically defined or not.

Within the context of an inflated flat we can readily show that there is a one-to-one correspondence between the (n-1)-dimensional subflats of Φ and the *n*-dimensional subspaces of Φ_+ . Moreover, *n* points of Φ affine-span one of its (n-1)-dimensional subflats if and only if every set of vectors which represents those points is an independent set in Φ_+ . (The concept of affine span of points is generalized through use of the span of the representing vectors to include all points, not just "finite" ones.)

The criterion for independence provided by exterior algebra (Corollary 65) leads to the conclusion that a nonzero exterior e-product (a **blade**) represents a subspace of Φ_+ and therefore also represents a subflat of Φ in the same homogeneous fashion that a single nonzero vector represents a point. That is, if $\{v_1, \ldots, v_n\}$ is an independent n-element set of vectors in Φ_+ , then the n-blade $v_1 \wedge \cdots \wedge v_n$ homogeneously represents an (n-1)-dimensional subflat of Φ . This is based on the following result.

Proposition 88 If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are independent n-element sets of vectors that span the same subspace, then the blades $w_1 \wedge \cdots \wedge w_n$ and $v_1 \wedge \cdots \wedge v_n$ are proportional, and conversely, if $w_1 \wedge \cdots \wedge w_n$ and $v_1 \wedge \cdots \wedge v_n$ are proportional blades made up of the vectors of the independent sets $\{w_1, \ldots, w_n\}$ and $\{v_1, \ldots, v_n\}$, then those sets of vectors span the same subspace.

Proof: Suppose that the independent sets $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of n vectors span the same subspace. Then each w_i can be expressed as a linear combination of $\{v_1, \ldots, v_n\}$. Putting these linear combinations in the blade $w_1 \wedge \cdots \wedge w_n$, after expanding and collecting the nonzero terms we are left with a nonzero (since $w_1 \wedge \cdots \wedge w_n$ is nonzero) multiple of the blade $v_1 \wedge \cdots \wedge v_n$.

On the other hand, suppose that $w_1 \wedge \cdots \wedge w_n$ and $v_1 \wedge \cdots \wedge v_n$ are proportional blades expressed as exterior products of vectors. Then

$$0 = w_i \wedge w_1 \wedge \cdots \wedge w_n = w_i \wedge v_1 \wedge \cdots \wedge v_n$$

so that each w_i is a linear combination of $\{v_1, \ldots, v_n\}$.

We will find it convenient to class the empty exterior product, which we by convention set equal to 1, and all its nonzero multiples, as blades that represents the empty flat and the subspace $\{0\}$. Thus, there is a one-to-one correspondence between the sets of proportional blades in $\bigwedge \Phi_+$ and the finite-dimensional subflats of Φ . Given the two blades $v_1 \wedge \cdots \wedge v_k$ and $v_{k+1} \wedge \cdots \wedge v_n$, then either $v_1 \wedge \cdots \wedge v_n = 0$ and the two corresponding subflats intersect, or $v_1 \wedge \cdots \wedge v_n$ represents the (generalized) affine sum of the two non-intersecting subflats. By the same token, given the blade β , the subspace of Φ_+ which it represents is $\{v \in \Phi_+ \mid v \wedge \beta = 0\}$. A blade that represents a hyperplane will be referred to as a **hyperplane blade**, and similarly, the terms **plane blade**, **line blade**, and **point blade** will be used.

8.4 Homogeneous and Plücker Coordinates

We suppose throughout this section and for the remainder of this chapter that Φ_+ has finite dimension d. Fix a basis \mathcal{B} for Φ_+ . The coordinates of a nonzero vector v of Φ_+ are known as **homogenous coordinates** for the point represented by v. Bases \mathcal{B}^{\wedge} for $\bigwedge \Phi_+$ may be obtained by choosing blades that are products of elements of \mathcal{B} . The coefficients of the expansion for the n-blade β in terms of the n-blades of any such \mathcal{B}^{\wedge} are known, in honor of German geometer Julius Plücker (1801-1868), as **Plücker coordinates** for the (n-1)-dimensional subflat of Φ represented by β , and sometimes, by abuse of language, as Plücker coordinates for any blade proportional to β . Homogenous coordinates are then Plücker coordinates for a point. Plücker coordinates are sometimes called **Grassmann coordinates** for German schoolteacher Hermann Grassmann (1809-1877), the first author to systematically treat vector spaces and vector algebras.

The coefficients of any linear combination of \mathcal{B} are homogenous coordinates for some point of Φ , but an arbitrary linear combination of the n-blades of a \mathcal{B}^{\wedge} cannot be guaranteed in general to yield Plücker coordinates for some

(n-1)-dimensional subflat of Φ . That is, it is not true in general that every nonzero element of $\bigwedge^n \Phi_+$ is expressible as an *n*-blade. We shall find, however, that every nonzero element of $\bigwedge^{d-1} \Phi_+$ is a (d-1)-blade.

Plücker coordinates of a blade may be expressed in terms of the coordinates of the vectors that make up the blade by expanding out the exterior product. If we suppose that we are given the n (possibly dependent) vectors v_1, \ldots, v_n with each v_j given in terms of basis vectors x_i by

$$v_j = \sum_{i=1}^d a_{i,j} \cdot x_i,$$

then

$$v_1 \wedge \dots \wedge v_n = \sum_{i_1=1}^d \dots \sum_{i_n=1}^d a_{i_1,1} \dots a_{i_n,n} \cdot x_{i_1} \wedge \dots \wedge x_{i_n}$$

and collecting terms on a particular set of basis elements (those that have subscripts that increase from left to right) we then get

$$\sum_{1 \leqslant i_1 < \dots < i_n \leqslant d} \left(\sum_{\sigma} \left(-1 \right)^{\sigma} a_{\sigma(i_1), 1} \cdots a_{\sigma(i_n), n} \right) \cdot x_{i_1} \wedge \dots \wedge x_{i_n}$$

where the inner sum is over all permutations

$$\sigma = \begin{pmatrix} i_1 & \cdots & i_n \\ \sigma(i_1) & \cdots & \sigma(i_n) \end{pmatrix}$$

of each subscript combination that the outer summation has selected. We recognize the inner sum as a determinant and thus the expansion may be written as

$$\sum_{1 \leqslant i_1 < \dots < i_n \leqslant d} \begin{vmatrix} a_{i_1,1} & \dots & a_{i_1,n} \\ \vdots & \dots & \vdots \\ a_{i_n,1} & \dots & a_{i_n,n} \end{vmatrix} \cdot x_{i_1} \wedge \dots \wedge x_{i_n} .$$

We may phrase this result in terms of points and infer the results of the following exercise which also contains a well-known result about the independence of the columns of a $d \times n$ matrix and the determinants of its $n \times n$ minors.

Exercise 8.2 Let P_1, \ldots, P_n be $n \leq d$ distinct points of Φ and for each j let $A_j = (a_{1,j}, \ldots, a_{d,j})$ be a d-tuple of homogenous coordinates for P_j in terms of some basis \mathcal{B} for Φ_+ . Let $A = [a_{i,j}]$ be the $d \times n$ matrix formed by using the A_j as its columns. Then if $\{P_1, \ldots, P_n\}$ is affine independent, there is a \mathcal{B}^{\wedge} such that in terms of its n-blades the $\binom{d}{n}$ determinants of $n \times n$ submatrices of A formed in each possible way by deleting all but n rows of A are Plücker coordinates for the affine span of $\{P_1, \ldots, P_n\}$. On the other hand, if $\{P_1, \ldots, P_n\}$ is affine dependent, these determinants all vanish.

8.5 Dual Description Via Annihilators

Remembering that we are assuming that Φ_+ has finite dimension d, Φ_+^{\top} also has finite dimension d. Each subspace \mathcal{W} of dimension n in Φ_+ has the annihilator \mathcal{W}^0 (the set of elements of Φ_+^{\top} which annihilate every vector of \mathcal{W}) that is a (d-n)-dimensional subspace of Φ_+^{\top} . As in Chapter 3, we identify $(\Phi_+^{\top})^{\top}$ with Φ_+ , and thereby are entitled to write $\mathcal{W}^{00} = (\mathcal{W}^0)^0 = \mathcal{W}$. The assignment of \mathcal{W}^0 to \mathcal{W} clearly creates a one-to-one correspondence between the subspaces of Φ_+ and the subspaces of Φ_+^{\top} , so that giving \mathcal{W}^0 is equivalent to giving \mathcal{W} .

 $\mathcal{W} = \mathcal{W}^{00}$ describes \mathcal{W} as the annihilator of \mathcal{W}^0 , which amounts to saying that \mathcal{W} is the set of those vectors in Φ_+ which are annihilated by every element of \mathcal{W}^0 . This dual description gives \mathcal{W} as the intersection of all the hyperplanes that contain it. Verification of this is provided by the following exercise, once we observe that the hyperplanes through 0 in Φ_+ are precisely the kernels of the linear functionals on Φ_+ .

Exercise 8.3 Let W be a subspace of Φ_+ . Then W^0 is the set of all elements of Φ_+^{\top} whose kernel contains W. Let \mathcal{X} be the intersection of all the kernels of the elements of W^0 , i. e., the intersection of all hyperplanes containing W. Show that $W = \mathcal{X}$ by verifying that $W \subset \mathcal{X}$ and $\mathcal{X} \subset W^{00}$.

We now point out and justify what is from a geometric standpoint perhaps the most important result concerning the annihilator, namely that for any subspaces W and X of Φ_+ we have

$$(\mathcal{W} + \mathcal{X})^0 = \mathcal{W}^0 \cap \mathcal{X}^0.$$

This is the same as saying that $f \in \Phi_+^{\top}$ satisfies f(w+x) = 0 for all $w \in \mathcal{W}$ and for all $x \in \mathcal{X}$ if and only if f(w) = 0 for all $w \in \mathcal{W}$ and f(x) = 0

for all $x \in \mathcal{X}$. The truth of the "if" part is obvious, and that of the "only if" part is nearly so. For, since 0 is in any subspace, it follows at once that f(w+0) = 0 for all $w \in \mathcal{W}$ and f(0+x) = 0 for all $x \in \mathcal{X}$. Interchanging Φ_+ with Φ_+^{\top} then gives us also

$$\left(\mathcal{W}^0+\mathcal{X}^0\right)^0=\mathcal{W}\cap\mathcal{X}$$
.

Exercise 8.4 Let f_1, \ldots, f_n be elements of Φ_+^{\top} . Then

$$(\{f_1,\ldots,f_n\})^0 = ((\{f_1\}) + \cdots + (\{f_n\}))^0 = \{f_1\}^0 \cap \cdots \cap \{f_n\}^0,$$

and thus any subspace of Φ_+ is the intersection of finitely many hyperplanes.

8.6 Direct and Dual Plücker Coordinates

Let $\{t_1, \ldots, t_n\}$ be a basis for the subspace \mathcal{W} of Φ_+ , and let $\mathcal{A} = \{t_1, \ldots, t_d\}$ be any extension of this basis for \mathcal{W} to a basis for Φ_+ . Then the basis \mathcal{A} has the dual $\mathcal{A}^{\top} = \{t_1^{\top}, \ldots, t_d^{\top}\}$, and $\{t_{n+1}^{\top}, \ldots, t_d^{\top}\}$ is a basis for \mathcal{W}^0 . Thus we have the blades proportional to $\omega = t_1 \wedge \cdots \wedge t_n$ representing \mathcal{W} , and the corresponding **annihilator blades** proportional to $\omega^0 = t_{n+1}^{\top} \wedge \cdots \wedge t_d^{\top}$ representing \mathcal{W}^0 (and therefore also representing \mathcal{W} in a dual sense).

Exercise 8.5 Given ω and ω^0 as above, the subspace of Φ_+ which ω represents is the set of all vectors $v \in \Phi_+$ that simultaneously satisfy the d-n equations $t_{n+1}^{\top}(v) = 0, \ldots, t_d^{\top}(v) = 0$.

For coordinatization purposes, fix a basis $\mathcal{B} = \{x_1, \dots, x_d\}$ for Φ_+ . Construct a basis \mathcal{B}^{\wedge} for $\bigwedge \Phi_+$ as we did previously, and similarly construct a basis $\mathcal{B}^{\top \wedge}$ for $\bigwedge \Phi_+^{\top}$ based on the elements of the dual basis \mathcal{B}^{\top} . In terms of \mathcal{B}^{\wedge} , Plücker coordinates stemming from blades proportional to ω will now be called **direct (Plücker) coordinates**, while coordinates in terms of $\mathcal{B}^{\top \wedge}$ and stemming from blades proportional to ω^0 will be called **dual (Plücker) coordinates**.

Each n-blade χ of \mathcal{B}^{\wedge} has a matching annihilator blade $\chi^{0} \in \mathcal{B}^{\top \wedge}$ which is the product of the d-n coordinate functions of the basis vectors not in χ . This gives a one-to-one match between the basis elements of \mathcal{B}^{\wedge} and $\mathcal{B}^{\top \wedge}$. We will find that, apart from an easily-determined sign factor, direct and dual coordinates corresponding to matching basis elements may be taken to be equal. Using the bases \mathcal{B}^{\wedge} and $\mathcal{B}^{\top \wedge}$ therefore makes it easy to convert

back and forth between direct and dual coordinates since at worst we need only multiply by -1. To see how this works, let us suppose, as we did just above, that we have a blade $\omega = t_1 \wedge \cdots \wedge t_n$ and its annihilator blade $\omega^0 = t_{n+1}^\top \wedge \cdots \wedge t_d^\top$ where $\mathcal{A} = \{t_1, \ldots, t_d\}$ is a basis for Φ_+ . The bases \mathcal{A} and \mathcal{A}^\top are respectively related to the coordinatization bases \mathcal{B} and \mathcal{B}^\top by an automorphism f and (Section 3.5) its contragredient $f^{-\top} = (f^{-1})^\top$ according to

$$t_j = f\left(x_j\right)$$
 and $t_j^{\top} = f^{-\top}\left(x_j^{\top}\right)$.

Now suppose that f and f^{-1} are given in terms of the elements of the basis \mathcal{B} by

$$f(x_j) = \sum_{i=1}^{d} a_{i,j} \cdot x_i$$
 and $f^{-1}(x_j) = \sum_{i=1}^{d} \alpha_{i,j} \cdot x_i$

and thus (Exercise 3.3)

$$f^{-\top}\left(x_{j}^{\top}\right) = \sum_{i=1}^{d} \alpha_{j,i} \cdot x_{i}^{\top}.$$

Therefore

$$\omega = t_1 \wedge \dots \wedge t_n = \left(\sum_{i=1}^d a_{i,1} \cdot x_i\right) \wedge \dots \wedge \left(\sum_{i=1}^d a_{i,n} \cdot x_i\right)$$

and

$$\omega^0 = t_{n+1}^\top \wedge \dots \wedge t_d^\top = \left(\sum_{i=1}^d \alpha_{n+1,i} \cdot x_i^\top\right) \wedge \dots \wedge \left(\sum_{i=1}^d \alpha_{d,i} \cdot x_i^\top\right).$$

For distinct i_1, \ldots, i_n we wish to compare a coordinate p_{i_1,\ldots,i_n} related to the basis element $x_{i_1} \wedge \cdots \wedge x_{i_n}$ in ω to that of a dual coordinate π_{i_{n+1},\ldots,i_d} related to the basis element $x_{i_{n+1}}^{\top} \wedge \cdots \wedge x_{i_d}^{\top}$ in ω^0 , where the two sets of subscripts are complements in $\{1,\ldots,d\}$. As our coordinates we take the coefficients of the relevant blades in the expressions for ω and ω^0 above, namely the determinants

$$p_{i_1,\dots,i_n} = \sum_{\sigma} (-1)^{\sigma} a_{\sigma(i_1),1} \cdots a_{\sigma(i_n),n}$$

and

$$\pi_{i_{n+1},\dots,i_d} = \sum_{\sigma} (-1)^{\sigma} \alpha_{n+1,\sigma(i_{n+1})} \cdots \alpha_{d,\sigma(i_d)}$$

where each sum is over all permutations σ of the indicated subscript set, and as usual, $(-1)^{\sigma} = +1$ or -1 according as the permutation σ is even or odd.

The following result attributed to the noted German mathematician Carl G. J. Jacobi (1804-1851) provides the final key.

Lemma 89 (Jacobi's Determinant Identity) Fix a basis $\mathcal{B} = \{x_1, \dots, x_d\}$ for Φ_+ and let g be the automorphism of Φ_+ such that for each j

$$g(x_j) = \sum_{i=1}^{d} b_{i,j} \cdot x_i$$
 and $g^{-1}(x_j) = \sum_{i=1}^{d} \beta_{i,j} \cdot x_i$.

Then

$$\det [c_{i,j}] = (\det g) \det [\gamma_{i,j}]$$

where $[c_{i,j}]$ is the $n \times n$ matrix with elements $c_{i,j} = b_{i,j}$, $1 \le i, j \le n$, and $[\gamma_{i,j}]$ is the $(d-n) \times (d-n)$ matrix with elements $\gamma_{i,j} = \beta_{i+n,j+n}$, $1 \le i, j \le d-n$.

Proof: Define the self map h of Φ_+ by specifying its values on \mathcal{B} as follows:

$$h(x_j) = \begin{cases} x_j, & j = 1, ..., n, \\ g^{-1}(x_j), & j = n + 1, ..., d. \end{cases}$$

Then

$$g(h(x_j)) = \begin{cases} g(x_j), & j = 1, \dots, n, \\ x_j, & j = n+1, \dots, d. \end{cases}$$

We have

$$x_1 \wedge \cdots \wedge x_n \wedge g^{-1}(x_{n+1}) \wedge \cdots \wedge g^{-1}(x_d) = (\det h) \cdot x_1 \wedge \cdots \wedge x_d$$

and it is readily established that $\det h = \det [\gamma_{i,j}]$. Similarly, $\det g \circ h = \det [c_{i,j}]$. The lemma now follows from the Product Theorem (Theorem 67).

We now construct a g that we will use in the lemma to produce the result we seek. Let ρ be the permutation such that $\rho(k) = i_k$. Define the automorphism q of Φ_+ by

$$q(x_i) = x_{\rho^{-1}(i)}.$$

We apply the lemma to $g = q \circ f$ so that

$$g(x_j) = q(f(x_j)) = \sum_{i=1}^d a_{i,j} \cdot q(x_i)$$

$$= \sum_{i=1}^d a_{i,j} \cdot x_{\rho^{-1}(i)} = \sum_{k=1}^d a_{\rho(k),j} \cdot x_k$$

$$= \sum_{k=1}^d a_{i_k,j} \cdot x_k = \sum_{k=1}^d b_{k,j} \cdot x_k$$

and

$$g^{-1}(x_j) = f^{-1}(q^{-1}(x_j)) = f^{-1}(x_{\rho(j)}) = \sum_{i=1}^d \alpha_{i,\rho(j)} \cdot x_i$$
$$= \sum_{i=1}^d \alpha_{i,i_j} \cdot x_i = \sum_{i=1}^d \beta_{i,j} \cdot x_i.$$

We observe that for this particular g

$$\det[c_{i,j}] = p_{i_1,\dots,i_n}, \ \det[\gamma_{i,j}] = \pi_{i_{n+1},\dots,i_d}.$$

We also see that

$$\det g = (\det q) (\det f) = (-1)^{\rho} \det f.$$

Hence, by the lemma

$$p_{i_1,\dots,i_n} = (-1)^{\rho} (\det f) \pi_{i_{n+1},\dots,i_d}$$

and this is the result we have been seeking. Up to the factor $(-1)^{\rho}$, ω and ω^0 may then be taken to have the same coordinate with respect to matching basis elements, since the factor $\det f$ may be ignored due to homogeneity.

Thus we may easily obtain a corresponding annihilator blade for any blade that is expressed as a linear combination of basis blades, and we then have the means to define a vector space isomorphism that sends each blade in $\bigwedge \Phi_+$ to a corresponding annihilator blade in $\bigwedge \Phi_+^{\mathsf{T}}$, as we now record.

Theorem 90 Let Φ_+ have the basis $\mathcal{B} = \{x_1, \dots, x_d\}$. Choose a basis \mathcal{B}^{\wedge} for $\bigwedge \Phi_+$ made up of exterior products (including the empty product which we take to equal 1) of the elements of \mathcal{B} . Similarly choose a basis $\mathcal{B}^{\top \wedge}$ for $\bigwedge \Phi_+^{\top}$ made up of exterior products of the elements of \mathcal{B}^{\top} . Let $H : \bigwedge \Phi_+ \to \bigwedge \Phi_+^{\top}$ be the vector space map such that for each $x_{i_1} \wedge \cdots \wedge x_{i_n} \in \mathcal{B}^{\wedge}$

$$H\left(x_{i_1}\wedge\cdots\wedge x_{i_n}\right)=(-1)^{\rho}\cdot x_{i_{n+1}}^{\top}\wedge\cdots\wedge x_{i_d}^{\top}$$

where $x_{i_{n+1}}^{\top} \wedge \cdots \wedge x_{i_d}^{\top} \in \mathcal{B}^{\top \wedge}$, $\{i_1, \dots, i_n\} \cup \{i_{n+1}, \dots, i_d\} = \{1, \dots, d\}$, and

$$\rho = \left(\begin{array}{ccc} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{array}\right).$$

Then H is a vector space isomorphism that sends each blade to a corresponding annihilator blade. \blacksquare

Exercise 8.6 The H of the theorem above does not depend on the particular order of the factors used to make up each element of the chosen bases \mathcal{B}^{\wedge} and $\mathcal{B}^{\top \wedge}$. That is, for any n and any permutation

$$\rho = \left(\begin{array}{ccc} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{array}\right)$$

we have

$$H\left(x_{i_1}\wedge\cdots\wedge x_{i_n}\right)=(-1)^{\rho}\cdot x_{i_{n+1}}^{\top}\wedge\cdots\wedge x_{i_d}^{\top}$$

and therefore

$$H^{-1}(x_{i_{n+1}}^{\top} \wedge \cdots \wedge x_{i_d}^{\top}) = (-1)^{\rho} \cdot x_{i_1} \wedge \cdots \wedge x_{i_n}.$$

Henceforth, unless stated otherwise, we suppose that we are employing the H based on a fixed underlying basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ with a fixed assignment of subscript labels to basis vectors.

Exercise 8.7 Use H and H^{-1} to show that any linear combination of a set of (d-1)-blades equals a (d-1)-blade or 0.

Exercise 8.8 The hyperplane blade $x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_d$ represents the subspace $\{x_i^{\top}\}^0$ and the corresponding subflat, called a **coordinate hyperplane**, for which the coordinate value corresponding to the basis vector x_i is always 0.

8.7 A Dual Exterior Product

A dual to our original "wedge" product \wedge on $\bigwedge \Phi_+$ is provided by the "vee" product \vee that we define by

$$\chi \vee \psi = H^{-1} \left(H \left(\chi \right) \wedge H \left(\psi \right) \right).$$

We may describe \vee as being obtained by "pulling back" \wedge through H.

Exercise 8.9 Verify that $\bigwedge \Phi_+$ with the \vee product is a legitimate vector algebra by showing that it is just an alias of $\bigwedge \Phi_+^{\top}$ with the \wedge product, obtained by relabeling ζ and η of $\bigwedge \Phi_+^{\top}$ as $H^{-1}(\zeta)$ and $H^{-1}(\eta)$, and writing \vee instead of \wedge . What is the unit element of this new algebra on $\bigwedge \Phi_+$?

The vector space $\bigwedge \Phi_+$ with the \vee product is thus another exterior algebra on the same space. In its dual relationship with the original exterior algebra, hyperplane blades play the rôle that point blades (nonzero vectors) play in the original. That is, the rôles of $\bigwedge^1 \Phi_+$ and $\bigwedge^{d-1} \Phi_+$ become interchanged, and in fact the rôles of $\bigwedge^n \Phi_+$ and $\bigwedge^{d-n} \Phi_+$ all become interchanged. The \vee product of independent hyperplane blades represents the intersection of the hyperplanes that those blades represent, just as the \wedge product of independent vectors represents the subspace sum of the subspaces that those vectors represent. The \vee product of dependent hyperplane blades is 0. The \vee product of two blades is 0 if and only if those blades represent subspaces that have a subspace sum smaller than the whole space Φ_+ , just as the \wedge product of two blades is 0 if and only if those blades represent subspaces that have an intersection larger than the 0 subspace. Following custom (and Grassmann's lead), we will call the original \wedge product **progressive** and the new \vee product regressive, terms that reflect the respective relationships of the products to subspace sum and intersection. In expressions, the progressive product will be given precedence so that by $u \wedge v \vee w \wedge x$ will be meant $(u \wedge v) \vee (w \wedge x)$.

Exercise 8.10 Let \bar{x}_i denote a hyperplane blade that represents $\left\{x_i^{\top}\right\}^0$. Then if $\{i_{k+1},\ldots,i_d\}=\{1,\ldots,d\}\setminus\{i_1,\ldots,i_k\}, \left\{x_{i_1}^{\top}\right\}^0\cap\cdots\cap\left\{x_{i_k}^{\top}\right\}^0$ is represented by $\bar{x}_{i_1}\vee\cdots\vee\bar{x}_{i_k}=\pm x_{i_{k+1}}\wedge\cdots\wedge x_{i_d}$.

8.8 Results Concerning Zero Coordinates

The basic significance of a zero coordinate is given in the following result.

Proposition 91 In the blade β , the Plücker coordinate corresponding to the basis element $\xi = x_{i_1} \wedge \cdots \wedge x_{i_n}$ is zero if and only if the flat represented by β intersects the flat represented by the blade $\bar{\xi} = x_{i_{n+1}} \wedge \cdots \wedge x_{i_d}$, where $\bar{I} = \{i_{n+1}, \ldots, i_d\}$ is the complement of $I = \{i_1, \ldots, i_n\}$ in $D = \{1, \ldots, d\}$.

Proof: $\psi \wedge \bar{\xi} = 0$ for each basis *n*-blade $\psi = x_{k_1} \wedge \cdots \wedge x_{k_n}$ with the exception of ξ . (Each *n*-element subset of D except I meets \bar{I} since the only *n*-element subset of I is I itself). Therefore $\beta \wedge \bar{\xi}$ is zero or not according as β 's Plücker coordinate corresponding to the basis element ξ is zero or not.

Exercise 8.11 The proposition above implies, as expected, that the point represented by the basis vector x_1 lies in the coordinate hyperplanes represented by $\{x_i^{\top}\}^0$, i = 2, ..., d.

The previous result serves as a prelude to the following generalization that treats the case where there is a particular family of zero coordinates.

Proposition 92 Let $1 \leq k \leq n \leq d$. The flat represented by the subspace $\mathcal{X} = \left\{x_{i_1}^{\top}\right\}^0 \cap \cdots \cap \left\{x_{i_k}^{\top}\right\}^0$ intersects the flat represented by the n-dimensional subspace \mathcal{Y} in a flat of dimension at least n-k if and only if the flat represented by \mathcal{Y} has a zero Plücker coordinate corresponding to each of the basis elements $x_{j_1} \wedge \cdots \wedge x_{j_n}$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$.

Proof: Suppose first that the flat represented by \mathcal{Y} has a zero Plücker coordinate corresponding to each of the basis elements $x_{j_1} \wedge \cdots \wedge x_{j_n}$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$. The flats represented by \mathcal{X} and \mathcal{Y} do intersect since by the previous proposition the flat represented by a subspace $\{x_{j_1}^\top\}^0 \cap \cdots \cap \{x_{j_n}^\top\}^0$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$ and must therefore intersect the flat represented by its factor \mathcal{X} . Denote by \bar{x}_m the basis element that represents the hyperplane $\{x_m^\top\}^0$. Then \mathcal{X} is represented by the regressive product $\xi = \bar{x}_{i_1} \vee \cdots \vee \bar{x}_{i_k}$, and each basis n-blade $x_{j_1} \wedge \cdots \wedge x_{j_n}$ is a regressive product of the d-n hyperplane blades $\bar{x}_{m_{n+1}}, \ldots, \bar{x}_{m_d}$ such that $m_{n+1}, \ldots, m_d \in \{1, \ldots, d\} \setminus \{j_1, \ldots, j_n\}$. Hence the regressive product $\eta \vee \xi$, where η is a blade that represents the flat that \mathcal{Y} represents, consists of a sum of terms of the form

$$p_{j_1,\dots,j_n}\cdot \bar{x}_{m_{n+1}}\vee\dots\vee \bar{x}_{m_d}\vee \bar{x}_{i_1}\vee\dots\vee \bar{x}_{i_k}$$

and upon careful scrutiny of these terms we find that each is zero, but only because the flat represented by \mathcal{Y} has a zero Plücker coordinate corresponding to each of the basis elements $x_{j_1} \wedge \cdots \wedge x_{j_n}$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$. The terms such that $\{j_1, \ldots, j_n\}$ excludes at least one element of $\{i_1, \ldots, i_k\}$ have each excluded i appearing as an m, while the terms such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$ have no i appearing as an m. Hence $\eta \vee \xi = 0$ if and only if the flat represented by \mathcal{Y} has a zero Plücker coordinate corresponding to each of the basis elements $x_{j_1} \wedge \cdots \wedge x_{j_n}$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$. But $\eta \vee \xi = 0$ if and only if the dimension of $\mathcal{X} + \mathcal{Y}$ is strictly less than d. The dimension of \mathcal{X} is d - k and the dimension of \mathcal{Y} is n so that by Grassmann's Relation (Corollary 37)

$$(d-k) + n - \dim(\mathcal{X} \cap \mathcal{Y}) = \dim(\mathcal{X} + \mathcal{Y}) < d \tag{*}$$

and hence

$$n-k < \dim (\mathcal{X} \cap \mathcal{Y})$$
.

The flat represented by $\mathcal{X} \cap \mathcal{Y}$ thus has dimension at least n-k.

Suppose on the other hand that \mathcal{X} and \mathcal{Y} intersect in a flat of dimension at least n-k. Then reversing the steps above, we recover Equation (*) so that $\eta \vee \xi = 0$ which can hold only if the flat represented by \mathcal{Y} has a zero Plücker coordinate corresponding to each of the basis elements $x_{j_1} \wedge \cdots \wedge x_{j_n}$ such that $\{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_n\}$.

Exercise 8.12 From Grassmann's Relation alone, show that the subspaces \mathcal{X} and \mathcal{Y} of the proposition above always intersect in a subspace of dimension at least n-k. Hence, without requiring any zero Plücker coordinates, we get the result that the flats represented by \mathcal{X} and \mathcal{Y} must intersect in a flat of dimension at least n-k-1.

8.9 Some Examples in Space

Let our flat of interest be $\Phi = \mathcal{F}^3$ for some field \mathcal{F} so that Φ_+ is a vector space of dimension d = 4. The basis \mathcal{B} will be $\{x_1, x_2, x_3, x_4\}$ where $\{x_2, x_3, x_4\}$ is a basis for Φ_0 and $\Phi_1 = x_1 + \Phi_0$ is Φ in its inflated context. In the following table we show how the elements of \mathcal{B}^{\wedge} and $\mathcal{B}^{\top \wedge}$ are connected by the isomorphism H, where we have only shown the subscript sequences (with

 \emptyset indicating the empty subscript sequence corresponding to the scalar 1).

We will generally use subscript sequences to describe the blades made up of basis vectors. Thus $3 \cdot 143$ will indicate either $3 \cdot x_1 \wedge x_4 \wedge x_3$ or $3 \cdot x_1^{\top} \wedge x_4^{\top} \wedge x_3^{\top}$ as dictated by context. In progressive products we will generally omit the \wedge sign, and in regressive products we will generally replace the \vee sign with a ".", so that (1+2)(1+3)=13+23 and $234.123=H^{-1}(-14)=-23$. We indicate the plane blade obtained by omitting any individual factor from 1234 by using the subscript of the omitted factor with a bar over it as in $234=\overline{1}, 134=\overline{2}$, etc. The elements of \mathcal{B}^{\wedge} may be expressed as regressive products of such plane blades as follows, where the empty regressive product of such plane blades is indicated by $\overline{\varnothing}$.

We shall often find it convenient to engage in a harmless abuse of language by referring to blades as the flats they represent.

We now present our first example. Consider the plane

$$(1+2)(1+3)(1+4) = 134 + 214 + 231 + 234 = \overline{2} - \overline{3} + \overline{4} + \overline{1}.$$

We wish to compute its intersection with the plane $\overline{2}$, which is then the line

$$(\overline{2} - \overline{3} + \overline{4} + \overline{1}) \cdot \overline{2} = -\overline{3} \cdot \overline{2} + \overline{4} \cdot \overline{2} + \overline{1} \cdot \overline{2} = -14 + 13 - 34.$$

We now have an answer in the form of a line blade, but we may wish also to know at least two points that determine the line. A point on a line can be obtained by taking its regressive product with any coordinate plane that does not contain it. The result of taking the regressive product of our line with each coordinate plane is as follows.

Thus the line pierces $\overline{1}$ (the plane at ∞) at 4-3, it pierces $\overline{3}$ at -1-4, and it pierces $\overline{4}$ at -1-3, but it is wholly contained in $\overline{2}$.

Let us now consider two lines in space. Let one be $\lambda=(1)\,(1+2+3+4)$ and the other be $\mu=(1+2)\,(1+3+4)$. They intersect because their progressive product is

$$(12+13+14)(13+14+21+23+24) = 1324+1423 = 0.$$

We can compute their affine sum by considering progressive products of one with the individual progressive point factors of the other. Multiplying 1 and μ gives $123+124=\overline{4}+\overline{3}$, the affine sum we seek. Multiplying 1+2+3+4 and μ also gives 123+124. The regressive product of the two lines is 0, confirming that their affine sum fails to be all of space. We can compute their intersection point by considering regressive products of one with the individual regressive plane factors of the other. We factor λ into plane blades:

$$\lambda = (12 + 13 + 14) = -(\overline{3}.\overline{4} + \overline{2}.\overline{4} + \overline{2}.\overline{3}) = -(\overline{3} + \overline{2}).(\overline{3} + \overline{4}).$$

 μ may be written as $\overline{2}.\overline{4}+\overline{2}.\overline{3}-\overline{3}.\overline{4}+\overline{1}.\overline{4}+\overline{1}.\overline{3}$ and taking its regressive product with $\overline{3}+\overline{2}$ gives

$$\overline{2}.\overline{4}.\overline{3} + \overline{1}.\overline{4}.\overline{3} - \overline{3}.\overline{4}.\overline{2} + \overline{1}.\overline{4}.\overline{2} + \overline{1}.\overline{3}.\overline{2} = 2 \cdot 1 + 2 + 3 + 4$$

whereas taking its regressive product with $\overline{3} + \overline{4}$ gives

$$\overline{2}.\overline{4}.\overline{3} + \overline{1}.\overline{4}.\overline{3} + \overline{2}.\overline{3}.\overline{4} + \overline{1}.\overline{3}.\overline{4} = 0.$$

 μ is therefore contained in the plane $\overline{3} + \overline{4}$ (as we already know since we found above that this same plane is the affine sum of λ and μ), but pierces the plane $\overline{3} + \overline{2}$ at the point $2 \cdot 1 + 2 + 3 + 4$ which must also be the point of intersection of the two lines.

Exercise 8.13 Find the intersection and the affine sum of the two lines $(\bar{1}) \cdot (\bar{1} + \bar{2} + \bar{3} + \bar{4})$ and $(\bar{1} + \bar{2}) \cdot (\bar{1} + \bar{3} + \bar{4})$.

Exercise 8.14 Suppose that the regressive product of two blades equals a nonzero scalar. What is the meaning of this both for the corresponding subspaces of Φ_+ and for the corresponding flats of Φ ? Give an example where the blades both are lines in space.

8.10 Factoring a Blade

As illustrated by the examples of the previous section, there are instances where we have a blade expressed in expanded form and we instead want to express it, up to a nonzero scalar factor, as the exterior product of vectors. That is, we seek an independent set of vectors that span the subspace represented by the blade. It is always possible to find such a factorization. One method for doing so will now be presented. We start by introducing the **extended coordinate array** of a blade based on a given set of Plücker coordinates for it. This is a full skew-symmetric array of values $P_{j_1,...,j_n}$ which contains the given set of Plücker coordinates along with either \pm duplicates of them or zeroes. This $d \times \cdots \times d$ (n factors of d) array is the generalization of a skew-symmetric $d \times d$ matrix. Given the Plücker coordinate $p_{i_1,...,i_n}$ related to the basis n-blade $x_{i_1} \wedge \cdots \wedge x_{i_n}$, the values $P_{j_1,...,j_n}$ at the array positions that correspond to this coordinate (the positions with subscripts that are the permutations of the subscripts of the given coordinate) are given by

$$P_{\sigma(i_1),\dots,\sigma(i_n)} = (-1)^{\sigma} p_{i_1,\dots,i_n}$$

for each permutation σ of i_1, \ldots, i_n . Entries in positions that have any equal subscripts are set to 0. The d-tuple of array values obtained by varying a given subscript in order from 1 to d, leaving all other subscripts fixed at chosen constants, is called a **file**, generalizing the concept of a row or column of a matrix. Any entry in the array is contained in exactly n files. Here now is the factorization result.

Proposition 93 (Blade Factorization) Given a set of Plücker coordinates for an n-blade, the n files that contain a given nonzero entry of the corresponding extended coordinate array constitute a factorization when we regard these files as vector coordinate d-tuples.

Proof: We may assume that the given Plücker coordinates were obtained as minor determinants in the manner described in Section 8.4 above. Using the notation of that section, each extended coordinate array value may be written in determinant form as

$$P_{j_1,\dots,j_n} = \begin{vmatrix} a_{j_1,1} & \cdots & a_{j_1,n} \\ \vdots & \cdots & \vdots \\ a_{j_n,1} & \cdots & a_{j_n,n} \end{vmatrix}.$$

We assume without loss of generality that the given nonzero entry is $P_{1,...,n}$. We will first show that each of the n files containing $P_{1,...,n}$ is in the span of the columns A_j of the matrix $A = [a_{i,j}]$, so that each is a coordinate d-tuple of a vector in the subspace represented by the blade with the given Plücker coordinates. Let the files containing $P_{1,...,n}$ form the columns B_j of the $d \times n$ matrix $B = [b_{i,j}]$ so that the file forming the column B_j has the elements

$$b_{i,j} = P_{1,\dots,j-1,i,j+1,\dots,n} = \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,1} & \cdots & a_{j-1,n} \\ a_{i,1} & \cdots & a_{i,n} \\ a_{j+1,1} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix} = \sum_{k=1}^{n} c_{j,k} a_{i,k},$$

where the last expression follows upon expanding the determinant about row j containing the $a_{i,k}$. Note that the coefficients $c_{j,k}$ are independent of i, and hence

$$B_j = \sum_{k=1}^n c_{j,k} A_k$$

showing that the file forming the column B_j is in the span of the columns of A.

To verify the independence of the vectors for which the B_j are coordinate d-tuples, let us examine the matrix B. Observe that the top n rows of B are $P_{1,\ldots,n}$ times the $n \times n$ identity matrix. Hence B contains an $n \times n$ minor with a nonzero determinant, and the vectors w_j for which the B_j are coordinate d-tuples must therefore be independent, since it follows from what we found in Section 8.4 that $w_1 \wedge \cdots \wedge w_n$ is not zero.

Exercise 8.15 For each nonzero extended coordinate, apply the method of the proposition above to the Plücker coordinates of the line obtained in the first example of the previous section. Characterize the sets of nonzero extended coordinates that are sufficient to use to yield all essentially different obtainable factorizations. Compare your results to the points that in the example were obtained by intersecting with coordinate planes.

Exercise 8.16 Use the method of the proposition above to factor the line λ of the second example of the previous section into the regressive product of plane blades. Do this first by factoring $H(\lambda)$ in $\bigwedge \Phi_+^{\top}$ and then carrying the result back to $\bigwedge \Phi_+$. Then also do it by factoring λ directly in the exterior algebra on $\bigwedge \Phi_+$ that the \vee product induces (where the "vectors" are the plane blades).

8.11 Algorithm for Sum or Intersection

Given the blades α and β , $\alpha \wedge \beta$ only represents their affine sum when the flats represented have an empty intersection. Otherwise the non-blade 0 is the result. Dually, $\alpha \vee \beta$ only represents their intersection when the flats represented have a full affine sum (all of Φ , that is). Otherwise the non-blade 0 is again the result. Blades that yield a 0 result when multiplied contain redundancy and pose for us the extra challenge of coping with that redundancy. As we have just seen in the previous section, a factorization for a blade is always readily obtainable, and we can exploit this to construct a resultant blade that omits redundant factors. We give an algorithm for doing this in the case where the affine sum is sought, and the same algorithm can be used on $H(\alpha)$ and $H(\beta)$ (or the similar \vee -based algorithm on α and β expressed in terms of basic hyperplane blades) when the intersection is sought. Suppose that β has been factored as

$$\beta = v_1 \wedge \dots \wedge v_n$$

and start off with $\gamma_0 = \alpha$. Then calculate successively $\gamma_1, \dots, \gamma_n$ according to

$$\gamma_i = \begin{cases} \gamma_{i-1} \wedge v_i & \text{if this is } \neq 0, \\ \gamma_{i-1} & \text{otherwise.} \end{cases}$$

Then γ_n is the desired blade that represents the affine sum. This algorithm is the obvious one based on omitting vectors that can already be expressed in

terms of the previously included ones that couldn't be so expressed. Which vectors are omitted, but not their number, depends on their labeling, of course. There are similarities here to the problems of finding the least common multiple and greatest common divisor of positive whole numbers, including that an algorithm seems to be required.

8.12 Adapted Coordinates and \vee Products

We continue to employ the fixed underlying basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ for the purposes of defining the isomorphism $H : \bigwedge \Phi_+ \to \bigwedge \Phi_+^{\mathsf{T}}$ and the resulting regressive product that it produces. However, simultaneously using other bases can sometimes prove useful, particularly in the case where we are dealing with a subspace of Φ_+ and we use coordinates adapted to a basis for that subspace. We will find that doing this will lead to a useful new view of the regressive product.

We start by considering finding coordinates for an n-blade γ that represents a subspace \mathcal{W} that is itself a subspace of the subspace \mathcal{U} of Φ_+ . We want these coordinates adapted to the basis $\{u_1, \ldots, u_l\}$ for the l-dimensional subspace \mathcal{U} . The way we will describe these coordinates involves some new notation.

Let $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, l\}$ and let $\overline{I} = \{i_{n+1}, \ldots, i_l\}$ denote its complement in $\{1, \ldots, l\}$. Note that this implies that $i_j = i_k$ only if j = k. Note also that the elements of both I and \overline{I} are shown as having subscript labels that imply choices have been made of an order for each in which their elements may be presented by following subscript numerical order. Thus we will treat I and \overline{I} as labeled sets that split $\{1, \ldots, l\}$ into two pieces (one of which could be empty, however). The order choices $i_1 < \cdots < i_n$ and $i_{n+1} < \cdots < i_l$ could be specified if one wishes. However, it will only be important that the relevant n-element subsets and their complements be given labelings that remain fixed throughout a particular discussion. The particular labelings will be unimportant. Based on this foundation, we now describe our new notation. We will denote the two permutations

$$\begin{pmatrix} 1 & \cdots & l \\ i_1 & \cdots & i_l \end{pmatrix}$$
 and $\begin{pmatrix} 1 & \cdots & l-n & l-n+1 & \cdots & l \\ i_{n+1} & \cdots & i_l & i_1 & \cdots & i_n \end{pmatrix}$

by $I\overline{I}$ and $\overline{I}I$ respectively. Also we will denote $u_{i_1} \wedge \cdots \wedge u_{i_n}$ by u_I , and $u_{i_{n+1}} \wedge \cdots \wedge u_{i_l}$ by $u_{\overline{I}}$.

Since γ is an *n*-blade that represents a subspace of the vector space \mathcal{U} that has the basis $\{u_1, \ldots, u_l\}$, we can expand γ in terms of the coordinates $(-1)^{\overline{I}I} a_I$ adapted to $\{u_1, \ldots, u_l\}$ as

$$\gamma = \sum_{I} \left(-1\right)^{\overline{I}I} a_{I} \cdot u_{I}$$

where the sum is over all subsets $I \subset \{1, ..., l\}$ such that |I| = n, where |I| denotes the number of elements of the set I. Each a_I is determined by

$$a_I \cdot u_1 \wedge \cdots \wedge u_l = u_{\overline{I}} \wedge \gamma$$

since if J is a particular one of the Is in the sum above for γ ,

$$u_{\overline{J}} \wedge \gamma = \sum_{I} (-1)^{\overline{I}I} a_{I} \cdot u_{\overline{J}} \wedge u_{I} = (-1)^{\overline{J}J} a_{J} \cdot u_{\overline{J}} \wedge u_{J} = a_{J} \cdot u_{1} \wedge \cdots \wedge u_{l}.$$

A similar result is obtained by using $(-1)^{I\overline{I}}$ instead of $(-1)^{\overline{I}I}$ in the expansion for γ .

We are now ready to begin to apply this to the intersection of a pair of subspaces in the case where that intersection corresponds to a nonzero regressive product. Thus let $\mathcal{U} = \langle \{u_1, \ldots, u_l\} \rangle$ and $\mathcal{V} = \langle \{v_1, \ldots, v_m\} \rangle$ be subspaces of Φ_+ of dimension l and m respectively. Let $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and let $\mathcal{U} + \mathcal{V} = \Phi_+$. Then by Grassmann's Relation (Corollary 37), \mathcal{W} has dimension n = l + m - d (so that, of course, $l + m \geqslant d$ since $n \geqslant 0$). Let γ be a blade that represents \mathcal{W} . Then the n-blade γ has the expansions

$$\gamma = \sum_{I \subset \{1, \dots, l\}} (-1)^{\overline{I}I} a_I \cdot u_I = \sum_{I \subset \{1, \dots, m\}} (-1)^{I\overline{I}} b_I \cdot v_I$$

where the one adapted to the vs is intentionally set up using $(-1)^{I\overline{I}}$ rather than $(-1)^{\overline{I}I}$, and the sums are over only those subsets I such that |I| = n. The a_I and b_I are determined by

$$a_I \cdot u_1 \wedge \cdots \wedge u_l = u_{\overline{I}} \wedge \gamma$$
 and $b_I \cdot v_1 \wedge \cdots \wedge v_m = \gamma \wedge v_{\overline{I}}$.

We can get some useful expressions for the a_I and b_I by first creating some blades that represent \mathcal{U} and \mathcal{V} and contain γ as a factor. To simplify our writing, denote $u_1 \wedge \cdots \wedge u_l$ by α and $v_1 \wedge \cdots \wedge v_m$ by β . Supposing that

 $\gamma = w_1 \wedge \cdots \wedge w_n$, the Replacement Theorem (Theorem 8) guarantees us blades $\widetilde{\alpha} = u_{j_1} \wedge \cdots \wedge u_{j_{l-n}}$ and $\widetilde{\beta} = v_{k_1} \wedge \cdots \wedge v_{k_{m-n}}$ such that for some nonzero scalars a and b we have $a \cdot \alpha = \widetilde{\alpha} \wedge \gamma$ and $b \cdot \beta = \gamma \wedge \widetilde{\beta}$. Note that $\widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta}$ is a d-blade because it is the product of vectors that span $\mathcal{U} + \mathcal{V} = \Phi_+$. We find then that

$$a_I \cdot \alpha \wedge \widetilde{\beta} = b \cdot u_{\overline{I}} \wedge \beta$$
 and $b_I \cdot \widetilde{\alpha} \wedge \beta = a \cdot \alpha \wedge v_{\overline{I}}$

and since

$$a \cdot \alpha \wedge \widetilde{\beta} = \widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta} = b \cdot \widetilde{\alpha} \wedge \beta$$

we then have

$$\frac{a_I}{ab} \cdot \widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta} = u_{\overline{I}} \wedge \beta \quad \text{and} \quad \frac{b_I}{ab} \cdot \widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta} = \alpha \wedge v_{\overline{I}}.$$

 $\widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta}$, $u_{\overline{I}} \wedge \beta$, and $\alpha \wedge v_{\overline{I}}$ are each the progressive product of d vectors and therefore are scalar multiples of the progressive product of all the vectors of any basis for Φ_+ such as the fixed underlying basis $\mathcal{B} = \{x_1, \dots, x_d\}$ that we use for the purposes of defining the isomorphism $H: \bigwedge \Phi_+ \to \bigwedge \Phi_+^{\top}$ and the resulting regressive product that it produces. We might as well also then use \mathcal{B} for extracting scalar multipliers in the above equations. We therefore define $[\xi]$, the **bracket** of the progressive product ξ of d vectors, via the equation $\xi = [\xi] \cdot x_1 \wedge \cdots \wedge x_d$. $[\xi]$ is zero when $\xi = 0$ and is nonzero otherwise (when ξ is a d-blade). Thus we have

$$a_I = \frac{ab}{\left[\widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta}\right]} \cdot \left[u_{\overline{I}} \wedge \beta\right] \quad \text{and} \quad b_I = \frac{ab}{\left[\widetilde{\alpha} \wedge \gamma \wedge \widetilde{\beta}\right]} \cdot \left[\alpha \wedge v_{\overline{I}}\right].$$

Dropping the factor that is independent of I, we find that we have the two equal expressions

$$\sum_{\substack{I \subset \{1,\dots,l\}\\|I|=n}} (-1)^{\overline{I}I} \left[u_{\overline{I}} \wedge \beta \right] \cdot u_I = \sum_{\substack{I \subset \{1,\dots,m\}\\|I|=n}} (-1)^{I\overline{I}} \left[\alpha \wedge v_{\overline{I}} \right] \cdot v_I \quad (**)$$

for an *n*-blade that represents $\mathcal{U} \cap \mathcal{V}$ and which therefore must be proportional to $\alpha \vee \beta$. The above development of (**) follows Garry Helzer's online class notes for *Math 431: Geometry for Computer Graphics* at the University of Maryland (see www.math.umd.edu/~gah/).

We note that, in agreement with what $\alpha \vee \beta$ gives, when the subspaces represented by α and β do not sum to all of Φ_+ , the two expressions of (**) both give 0. This is because the totality of the vectors that make up α and β then do not span Φ_+ and therefore neither do any d of them. Let us denote these equal expressions by $\mu(\alpha, \beta)$ in any case where the sum of the degrees of the blades α and β is at least d.

We know that $\mu(\alpha, \beta) = c \cdot \alpha \vee \beta$ but it remains to be seen how c depends on α and β . Let us see what happens in a simple example where α and β are made up of xs drawn from the underlying basis \mathcal{B} . Let

$$\alpha = x_{j_1} \wedge \cdots \wedge x_{j_l}$$
 and $\beta = x_{j_1} \wedge \cdots \wedge x_{j_n} \wedge x_{j_{l+1}} \wedge \cdots \wedge x_{j_d}$

where $\{x_{j_1}, \ldots, x_{j_d}\} = \mathcal{B}$. Using the right-hand expression of (**), we find that it gives the single nonzero term $(-1)^{\sigma} \cdot x_{j_1} \wedge \cdots \wedge x_{j_n}$ for $\mu(\alpha, \beta)$, where

$$\sigma = \left(\begin{array}{ccc} 1 & \cdots & d \\ j_1 & \cdots & j_d \end{array}\right).$$

We also find that

$$H(\alpha) = (-1)^{\sigma} \cdot x_{j_{l+1}}^{\top} \wedge \cdots \wedge x_{j_d}^{\top}$$
 and $H(\beta) = (-1)^{\rho} \cdot x_{j_{n+1}}^{\top} \wedge \cdots \wedge x_{j_l}^{\top}$

where

$$\rho = \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+d-l & n+d-l+1 & \cdots & d \\ j_1 & \cdots & j_n & j_{l+1} & \cdots & j_d & j_{n+1} & \cdots & j_l \end{pmatrix}$$

so that

$$H(\alpha) \wedge H(\beta) = (-1)^{\sigma} (-1)^{\rho} \cdot x_{j_{l+1}}^{\top} \wedge \dots \wedge x_{j_d}^{\top} \wedge x_{j_{n+1}}^{\top} \wedge \dots \wedge x_{j_l}^{\top}$$
$$= (-1)^{\sigma} \cdot H(x_{j_1} \wedge \dots \wedge x_{j_n})$$

and therefore $\alpha \vee \beta = (-1)^{\sigma} \cdot x_{j_1} \wedge \cdots \wedge x_{j_n} = \mu(\alpha, \beta)$. This example leads us to conjecture that it is always the case that $\mu(\alpha, \beta) = \alpha \vee \beta$, a result that we will soon prove.

Exercise 8.17 Verify the results of the example above in the case where d = 4, $u_1 = x_3, u_2 = x_2, u_3 = x_1$, and $v_1 = x_3, v_2 = x_2, v_3 = x_4$. (In evaluating the second expression of (**), be careful to base I and \overline{I} on the subscripts of the v_3 , not on the subscripts of the v_3 , and notice that σ comes from the bracket.)

While \vee is a bilinear function defined on all of $\bigwedge \Phi_+ \times \bigwedge \Phi_+$, μ is only defined for pairs of blades that have degrees that sum to at least d. Extending μ as a bilinear function defined on all of $\bigwedge \Phi_+ \times \bigwedge \Phi_+$ will allow us prove equality with \vee by considering only what happens to the elements of $\mathcal{B}^{\wedge} \times \mathcal{B}^{\wedge}$. We thus define the bilinear function $\widetilde{\mu} : \bigwedge \Phi_+ \times \bigwedge \Phi_+ \to \bigwedge \Phi_+$ by defining it on $\mathcal{B}^{\wedge} \times \mathcal{B}^{\wedge}$ as

$$\widetilde{\mu}(x_J, x_K) = \begin{cases} \mu(x_J, x_K) \text{ if } |J| + |K| \geqslant d, \\ 0 \text{ otherwise.} \end{cases}$$

where $J, K \subset \{1, ..., d\}$. We now show that when α and β are blades that have degrees that sum to at least d, $\widetilde{\mu}(\alpha, \beta) = \mu(\alpha, \beta)$, so that $\widetilde{\mu}$ extends μ . Suppose that

$$\alpha = \sum_{J} a_J \cdot x_J$$
 and $\beta = \sum_{K} b_K \cdot x_K$.

Then, since the bracket is clearly linear, using in turn each of the expressions of (**) which define μ , we find that

$$\mu(\alpha, \beta) = \sum_{I} (-1)^{\overline{I}I} \left[u_{\overline{I}} \wedge \sum_{K} b_{K} \cdot x_{K} \right] \cdot u_{I} = \sum_{K} b_{K} \cdot \mu(\alpha, x_{K}) =$$

$$= \sum_{K} b_{K} \cdot \sum_{J} a_{J} \cdot \mu(x_{J}, x_{K}) = \widetilde{\mu}(\alpha, \beta)$$

and $\widetilde{\mu}$ indeed extends μ . We now are ready to formally state and prove the conjectured result.

Theorem 94 For any $\eta, \zeta \in \bigwedge \Phi_+$, $\widetilde{\mu}(\eta, \zeta) = \eta \vee \zeta$. Hence for blades $\alpha = u_1 \wedge \cdots \wedge u_l$ and $\beta = v_1 \wedge \cdots \wedge v_m$, of respective degrees l and m such that $l + m \geqslant d$, we have

$$\alpha \vee \beta = \sum_{\substack{I \subset \{1, \dots, l\} \\ |I| = n}} \left(-1\right)^{\overline{I}I} \left[u_{\overline{I}} \wedge \beta \right] \cdot u_{I} = \sum_{\substack{I \subset \{1, \dots, m\} \\ |I| = n}} \left(-1\right)^{I\overline{I}} \left[\alpha \wedge v_{\overline{I}} \right] \cdot v_{I} \qquad (* \vee *)$$

where n = l + m - d.

Proof: We need only verify that $\widetilde{\mu}(\eta,\zeta) = \eta \vee \zeta$ for $(\eta,\zeta) \in \mathcal{B}^{\wedge} \times \mathcal{B}^{\wedge}$. Thus we assume that $(\eta,\zeta) \in \mathcal{B}^{\wedge} \times \mathcal{B}^{\wedge}$. Since $\widetilde{\mu}(\eta,\zeta) = \eta \vee \zeta = 0$ when the

subspaces represented by η and ζ do not sum to Φ_+ (including the case when the degrees of η and ζ sum to less than d), we assume that the subspaces represented by η and ζ do sum to Φ_+ (and therefore that the degrees of η and ζ sum to at least d). The cases where $\eta = \alpha$ and $\zeta = \beta$ where α and β are given as

$$\alpha = x_{j_1} \wedge \cdots \wedge x_{j_l}$$
 and $\beta = x_{j_1} \wedge \cdots \wedge x_{j_n} \wedge x_{j_{l+1}} \wedge \cdots \wedge x_{j_d}$

have been verified in the example above. The forms of α and β are sufficiently general that every case we need to verify is obtainable by separately permuting factors inside each of α and β . This merely results in $\eta = (-1)^{\tau} \cdot \alpha$ and $\zeta = (-1)^{v} \cdot \beta$ so that $\eta \vee \zeta = (-1)^{\tau} (-1)^{v} \cdot \alpha \vee \beta = \widetilde{\mu}(\eta, \zeta)$ by bilinearity.

The formulas $(* \vee *)$ of the theorem above express $\alpha \vee \beta$ in all the nontrivial cases, i. e., when the respective degrees l and m of the blades α and β sum to at least d, and they do it in a coordinate-free manner. This is not to say that the idea of coordinates (or bases) is missing from these formulas or their derivation. But these formulas do not involve a priori coordinates for α and β , or their factors, in terms of some overall basis. Coordinates appear in these formulas as outputs, not as inputs. We have defined the bracket in terms of the overall basis \mathcal{B} (the same one used to define H), and the bracket is a coordinate function. The derivation of these formulas proceeded by adapting coordinates to the blade factors, and then bringing in the bracket to express the adapted coordinates as functions of the blade factors. The result is a formulation of the regressive product where only the blade factors appear explicitly. These formulas are known to be important in symbolic algebraic computations, as opposed to the numeric computations that we have previously carried out using a priori coordinates based on \mathcal{B} . They also now render the following result readily apparent.

Corollary 95 Any two regressive products are proportional.

Proof: The bracket depends on the basis ${\mathcal B}$ only up to a nonzero factor.

8.13 Problems

- 1. Theorem 87 and its proof require that P be distinct from the other points. Broaden the theorem by giving a similar proof for the case where P coincides with A.
- 2. What is the effect on H of a change in the assignment of the subscript labels $1, \ldots, d$ to the vectors of \mathcal{B} ?
- 3. Determine the proportionality factor between two regressive products as the determinant of a vector space map.
- 4. Notating as in Section 8.9, 12+34 is not a blade in the exterior algebra of a 4-dimensional vector space.
 - 5. Referring to Theorem 90, for arbitrary vectors v_1, \ldots, v_n of $\Phi_+, n \leq d$,

$$H\left(v_{1} \wedge \cdots \wedge v_{n}\right) = \begin{vmatrix} x_{1}^{\top}\left(v_{1}\right) & \cdots & x_{1}^{\top}\left(v_{n}\right) & x_{1}^{\top}\left(\cdot_{1}\right) & \cdots & x_{1}^{\top}\left(\cdot_{d-n}\right) \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{n}^{\top}\left(v_{1}\right) & \cdots & x_{n}^{\top}\left(v_{n}\right) & x_{n}^{\top}\left(\cdot_{1}\right) & \cdots & x_{n}^{\top}\left(\cdot_{d-n}\right) \\ x_{n+1}^{\top}\left(v_{1}\right) & \cdots & x_{n+1}^{\top}\left(v_{n}\right) & x_{n+1}^{\top}\left(\cdot_{1}\right) & \cdots & x_{n+1}^{\top}\left(\cdot_{d-n}\right) \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{d}^{\top}\left(v_{1}\right) & \cdots & x_{d}^{\top}\left(v_{n}\right) & x_{d}^{\top}\left(\cdot_{1}\right) & \cdots & x_{d}^{\top}\left(\cdot_{d-n}\right) \end{vmatrix}$$

where $x_i^\top(\cdot_k)$ and $x_j^\top(\cdot_k)$ denote the unevaluated functionals x_i^\top and x_j^\top , each of which must be evaluated at the same k-th vector of some sequence of vectors. The resulting determinant is then a function of $x_1^\top, \cdots, x_d^\top$ each potentially evaluated once at each of d-n arguments, which we identify with a blade in $\bigwedge \Phi_+^\top$ via the isomorphism of Theorem 71.

9 Vector Projective Geometry

9.1 The Projective Structure on \mathcal{V}

Recall that in Chapter 6 we introduced the set V(V) of all subspaces of the vector space V, defined the affine structure on V as A(V) = V + V(V), and interpreted the result as an affine geometry based on V. We now take just V(V) by itself, call it the **projective structure** on V, and interpret it as a projective geometry based on V. The **projective flats** are thus the subspaces of V. The **points** are the one-dimensional subspaces of V, the **lines** are the two-dimensional subspaces of V, the **planes** are the three-dimensional subspaces of V, and the **hyperplanes** are the subspaces of V of codimension one. When $\dim V = 2$, the projective geometry based on V is called a **projective line**, when $\dim V = 3$, it is called a **projective plane**, and in the general finite-dimensional case when $\dim V = d$, it is called a **projective** (d-1)-space. The n-dimensional subspaces of V are said to have **projective dimension** n-1. Thus we introduce the function pdim defined by

$$pdim(\mathcal{X}) = dim(\mathcal{X}) - 1$$

to give the projective dimension of the flat or subspace \mathcal{X} . The trivial subspace $0 = \{0\}$ is called the **null flat** and is characterized by pdim (0) = -1. Grassmann's Relation (Corollary 37) may be reread as

$$\operatorname{pdim} \mathcal{X} + \operatorname{pdim} \mathcal{Y} = \operatorname{pdim}(\mathcal{X} + \mathcal{Y}) + \operatorname{pdim}(\mathcal{X} \cap \mathcal{Y}).$$

We also refer to a subspace sum as the **join** of the corresponding projective flats. We denote by $P(\mathcal{V})$ the set of all the points of the geometry and we put $P^+(\mathcal{V}) = P(\mathcal{V}) \cup 0$. All flats are joins of points, with the null flat being the empty join. If \mathcal{X} is a subspace of \mathcal{V} , then $V(\mathcal{X}) \subset V(\mathcal{V})$ gives us a projective geometry in its own right, a **subgeometry** of $V(\mathcal{V})$, with $P(\mathcal{X}) \subset P(\mathcal{V})$.

 \mathcal{V} is conceptually the same as our Φ_+ of the previous chapter, except that it is stripped of all references to the affine flat Φ . No points of $\mathsf{P}(\mathcal{V})$ are special. The directions that were viewed as special "points at infinity" in the viewpoint of the previous chapter become just ordinary points without any designated Φ_0 for them to inhabit. We will at times find it useful to designate some hyperplane through 0 in \mathcal{V} to serve as a Φ_0 and thereby allow \mathcal{V} to be interpreted in a generalized affine manner. However, independent of any generalized affine interpretation, we have the useful concept of homogeneous representation of points by vectors which we continue to exploit.

9.2 Projective Frames

Notice: Unless otherwise stated, we assume for the rest of this chapter that \mathcal{V} is a nontrivial vector space of finite dimension d over the field \mathcal{F} .

The points X_0, \ldots, X_n of $P(\mathcal{V})$ are said to be in **general position** if they are distinct and have vector representatives that are as independent as possible, i. e., either all of them are independent, or, if $n \ge d$, every d of them are independent. The case n = d is of particular interest: any d + 1 points in general position are said to form a **projective frame** for $V(\mathcal{V})$.

Exercise 9.1 Any 3 distinct points of a projective line form a projective frame for it.

To say that the d+1 distinct points X_0, \ldots, X_d are a projective frame for $V(\mathcal{V})$ is the same as to say that there are respective representing vectors x_0, \ldots, x_d such that $\{x_1, \ldots, x_d\}$ is an independent set (and therefore a basis for \mathcal{V}) and $x_0 = a_1 \cdot x_1 + \cdots + a_d \cdot x_d$, where all of the scalar coefficients a_i are nonzero. Given that $\{x_1, \ldots, x_d\}$ is a basis, then with the same nonzero $a_i, \{a_1 \cdot x_1, \ldots, a_d \cdot x_d\}$ is also a basis. Hence the following result.

Proposition 96 Points X_0, \ldots, X_d of a projective frame for $V(\mathcal{V})$ can be represented respectively by vectors x_0, \ldots, x_d such that $\{x_1, \ldots, x_d\}$ is a basis for \mathcal{V} and $x_0 = x_1 + \cdots + x_d$.

A representation of the points X_0, \ldots, X_d of a projective frame by respective vectors x_0, \ldots, x_d such that $x_0 = x_1 + \cdots + x_d$ is said to be **standardized**. In such a standardized representation, X_0 is called the **unit point**, and X_1, \ldots, X_d are known by various terms such as **base points**, **vertices of the simplex of reference**, or **fundamental points of the coordinate system**. In terms of coordinate d-tuples, any d+1 points X_0, \ldots, X_d in general position can thus be assigned the respective coordinate d-tuples $\varepsilon_0 = (1, 1, \ldots, 1)$, $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0)$, $\ldots, \varepsilon_d = (0, \ldots, 0, 1)$. However, up to the unavoidable (and ignorable) nonzero overall scalar multiplier, there is only one basis for $\mathcal V$ that permits this assignment.

Proposition 97 The basis $\{x_1, \ldots, x_d\}$ of the previous proposition is unique up to a nonzero overall scalar multiplier.

Proof: Suppose that X_1, \ldots, X_d are also represented respectively by the vectors y_1, \ldots, y_d which together automatically form a basis for \mathcal{V} due to the general position hypothesis, and X_0 is represented by $y_0 = y_1 + \cdots + y_d$. Then $y_i = a_i \cdot x_i$ for each i with a_i a nonzero scalar. Hence

$$y_0 = y_1 + \dots + y_d = a_1 \cdot x_1 + \dots + a_d \cdot x_d = a_0 \cdot (x_1 + \dots + x_d)$$

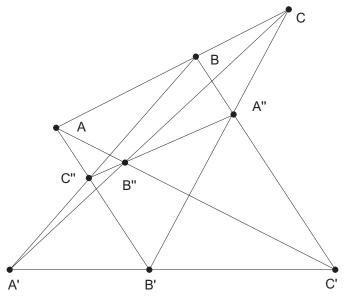
and since $\{x_1, \ldots, x_d\}$ is a basis for \mathcal{V} we must have $a_i = a_0$ for all i > 0. Hence $y_i = a_0 \cdot x_i$ for all i > 0 which is what was to be shown.

Designating a standardized representation of the points of a projective frame thus uniquely specifies a system of homogeneous coordinates on P(V). Given that homogeneous coordinate d-tuples for the unit point and the base points are ε_0 and $\varepsilon_1, \ldots, \varepsilon_d$, coordinate d-tuples of all the points of P(V) are completely determined in the homogeneous sense (i. e., up to a nonzero scalar multiplier).

9.3 Pappus Revisited

Employing a projective frame can ease the proof of certain theorems of projective geometry. The Theorem of Pappus is one such. We previously stated and proved one of its affine versions as Theorem 82. That proof employed rather elaborate barycentric calculations. The projective version that we now state and prove has essentially the same statement in words as Theorem 82 but now refers to points and lines in a projective plane over a suitable field.

Theorem 98 (Theorem of Pappus) Let l, l' be distinct coplanar lines with distinct points $A, B, C \subset l \setminus l'$ and distinct points $A', B', C' \subset l' \setminus l$. If BC' meets B'C in A'', AC' meets A'C in B'', and AB' meets A'B in C'', then A'', B'', C'' are distinct collinear points.



Illustrating the Theorem of Pappus

Proof: We choose the points B', A, B, C' as our projective frame. These points are in general position since any three of them always contain one that is not allowed to be on the line through the other two. We designate a standardized representation by choosing vectors x_1, x_2, x_3 to respectively represent A, B, C' and $x_0 = x_1 + x_2 + x_3$ to represent the unit point B'.

 $AB = \langle \{x_1, x_2\} \rangle$ and C on it can be represented as $x_1 + c \cdot x_2$, or what is the same, as the coordinate tuple (1, c, 0).

 $B'C' = \langle \{x_1 + x_2 + x_3, x_3\} \rangle$ and A' on it can be represented by (1, 1, a).

 $a \neq 0$ because letting a = 0 in the representation (1, 1, a) for A' would put A' on AC. This is so because $AC = \langle \{x_1, x_1 + c \cdot x_2\} \rangle$ contains $c \cdot (x_1 + x_2) = (c-1) \cdot x_1 + (x_1 + c \cdot x_2)$ and $c \neq 0$ since $C \neq A$. Similarly, $c \neq 1$ because letting c = 1 in the representation (1, c, 0) of C would put C on A'B'. This is so because $A'B' = \langle \{x_1 + x_2 + a \cdot x_3, x_1 + x_2 + x_3\} \rangle$ which contains $(1 - a) \cdot (x_1 + x_2) = (x_1 + x_2 + a \cdot x_3) - a \cdot (x_1 + x_2 + x_3)$ and $a \neq 1$ since $A' \neq B'$.

 $BC' = \langle \{x_2, x_3\} \rangle$, $B'C = \langle \{x_1 + x_2 + x_3, x_1 + c \cdot x_2\} \rangle$ and $(1 - c) \cdot x_2 + x_3 = (x_1 + x_2 + x_3) - (x_1 + c \cdot x_2)$ lies in both so that A'' is represented by (0, 1 - c, 1).

 $C'A = \langle \{x_1, x_3\} \rangle$, $A'C = \langle \{x_1 + x_2 + a \cdot x_3, x_1 + c \cdot x_2\} \rangle$ and $(1 - c) \cdot x_1 - ca \cdot x_3 = x_1 + c \cdot x_2 - c \cdot (x_1 + x_2 + a \cdot x_3)$ lies in both so that B'' is represented by (1 - c, 0, -ca).

 $AB' = \langle \{x_1, x_1 + x_2 + x_3\} \rangle$, $A'B = \langle \{x_1 + x_2 + a \cdot x_3, x_2\} \rangle$ and $x_1 + a \cdot x_2 + a \cdot x_3 = (1 - a) \cdot x_1 + a \cdot (x_1 + x_2 + x_3) = (x_1 + x_2 + a \cdot x_3) + (a - 1) \cdot x_2$ lies in both so that C'' is represented by (1, a, a).

The dependency relation

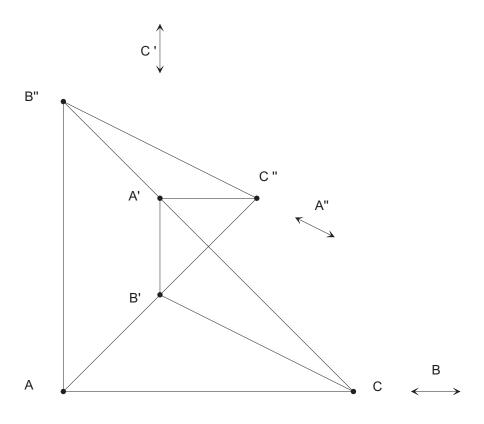
$$a \cdot (0, 1 - c, 1) + (1 - c, 0, -ca) + (c - 1) \cdot (1, a, a) = 0$$

shows that A'', B'', C'' are collinear. A'', B'', C'' are distinct since $a \neq 0$ and $c \neq 1$ (shown earlier in this proof) make it impossible for any one of (0, 1 - c, 1), (1 - c, 0, -ca), (1, a, a) to be a multiple of any other.

What we have just done in proving the Theorem of Pappus becomes even more transparent when viewed in the light of a generalized affine interpretation. Using the same basis as for our chosen standardized representation, and letting $x_1 = 1$ be our Φ_1 just as we did in the previous chapter, we get the generalized affine interpretation of a rather simple affine theorem. The flat Φ that we view as having been inflated to Φ_1 has point A at its origin, and the points B and C' have been "sent to infinity" in two different directions. As a result, A'' is at infinity, and the conclusion of the theorem is that B''C'' is parallel to B'C, as depicted in the figure below. The proof above is just a proof of this affine theorem using the inflated flat method. We prove exactly the same thing when we give an ordinary proof of the affine theorem, as we now do. In coordinate form, the finite points are then A = (0,0), C = (c,0), B' = (1,1), A' = (1,a), C'' = (a,a), $B'' = (0,ac(c-1)^{-1})$. Thus

$$C'' - B'' = (a, a(1-c)^{-1}) = a(1-c)^{-1} \cdot (1-c, 1) = a(1-c)^{-1} \cdot (B'-C)$$

so that B''C'' is parallel to B'C as was to be shown.



Generalized Affine Interpretation

9.4 Projective Transformations

Besides bases and frames, certain functions play an important rôle in projective geometry. Let \mathcal{W} be an alias of \mathcal{V} . Any vector space isomorphism $f: \mathcal{V} \to \mathcal{W}$ induces a bijection from $P(\mathcal{V})$ to $P(\mathcal{W})$. By a **projective transformation** we will mean any such bijection from $P(\mathcal{V})$ to $P(\mathcal{W})$ induced by a vector space isomorphism from \mathcal{V} to \mathcal{W} . The terms **projectivity** and **homography** are also commonly used. For any nonzero scalar a, f and $a \cdot f$ clearly induce the same projective transformation. On the other hand, if $g: \mathcal{V} \to \mathcal{W}$ is another vector space isomorphism that induces the same projective transformation as f, then $g = a \cdot f$ for some nonzero scalar a, as we now show. Let $\{x_1, \ldots, x_d\}$ be a basis for \mathcal{V} . For each i we must have $g(x_i) = a_i \cdot f(x_i)$ for some nonzero scalar a_i . Also

 $g(x_1 + \cdots + x_d) = a \cdot f(x_1 + \cdots + x_d)$ for some nonzero scalar a. But then

$$a_1 \cdot f(x_1) + \dots + a_d \cdot f(x_d) = a \cdot f(x_1) + \dots + a \cdot f(x_d)$$

and since $\{f(x_1), \ldots, f(x_d)\}$ is an independent set, $a_i = a$ for all i. Hence $g(x_i) = a \cdot f(x_i)$ for all i and therefore $g = a \cdot f$. Thus, similar to the classes of proportional vectors as the representative classes for points, we have the classes of proportional vector space isomorphisms as the representative classes for projective transformations. A vector space isomorphism homogeneously represents a projective transformation in the same fashion that a vector represents a point. For the record, we now formally state this result as the following theorem.

Theorem 99 Two isomorphisms between finite-dimensional vector spaces induce the same projective transformation if and only if these isomorphisms are proportional. ■

It is clear that projective transformations send projective frames to projective frames. Given an arbitrary projective frame for $V(\mathcal{V})$ and another for $V(\mathcal{W})$, there is an obvious projective transformation that sends the one to the other. This projective transformation, in fact, is uniquely determined.

Theorem 100 Let X_0, \ldots, X_d and Y_0, \ldots, Y_d be projective frames for V(V) and V(W), respectively. Then there is a unique projective transformation from P(V) to P(W) which for each i sends X_i to Y_i .

Proof: Let x_1, \ldots, x_d be representative vectors corresponding to X_1, \ldots, X_d and such that $x_0 = x_1 + \cdots + x_d$ represents X_0 . Similarly, let y_1, \ldots, y_d be representative vectors corresponding to Y_1, \ldots, Y_d and such that $y_0 = y_1 + \cdots + y_d$ represents Y_0 . Then the vector space map $f: \mathcal{V} \to \mathcal{W}$ that sends x_i to y_i for i > 0 induces a projective transformation that for each i sends X_i to Y_i . Suppose that the vector space map $g: \mathcal{V} \to \mathcal{W}$ also induces a projective transformation that for each i sends X_i to Y_i . Then there are nonzero scalars a_0, \ldots, a_d such that, for each i, $g(x_i) = a_i \cdot y_i$ so that then

$$g(x_0) = g(x_1 + \dots + x_d) = a_1 \cdot y_1 + \dots + a_d \cdot y_d = a_0 \cdot (y_1 + \dots + y_d)$$

and since $\{y_1, \ldots, y_d\}$ is an independent set, it must be that all the a_i are equal to a_0 . Hence $g = a_0 \cdot f$, so that f and g induce exactly the same projective transformation.

9.5 Projective Maps

Because P(W) does not contain the null flat, only a one-to-one vector space map from \mathcal{V} to some other vector space \mathcal{W} will induce a function from $P(\mathcal{V})$ to $P(\mathcal{W})$. However, any vector space map from \mathcal{V} to \mathcal{W} does induce a function from $P^+(\mathcal{V})$ to $P^+(\mathcal{W})$. By a **projective map** we will mean any function from $P^+(\mathcal{V})$ to $P^+(\mathcal{W})$ induced by a vector space map from \mathcal{V} to \mathcal{W} . Technically, a projective transformation is not a projective map because it is from $P(\mathcal{V})$ to $P(\mathcal{W})$, not from $P^+(\mathcal{V})$ to $P^+(\mathcal{W})$. However, each projective transformation clearly extends to a unique bijective projective map from $P^+(\mathcal{V})$ to $P^+(\mathcal{V})$.

Exercise 9.2 The composite of vector space maps induces the composite of the separately induced projective maps, and the identity induces the identity.

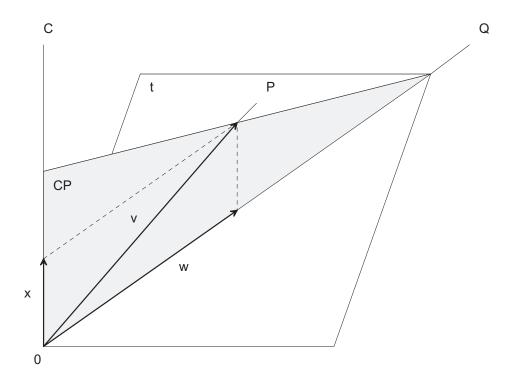
A vector space map from \mathcal{V} to \mathcal{W} also induces a function from $V(\mathcal{V})$ to $V(\mathcal{W})$ and we could just as well have used this as our projective map. A function is also induced from $P(\mathcal{V}) \setminus P(\mathcal{K})$ to $P(\mathcal{W})$, where \mathcal{K} is the kernel of the vector space map, and some authors call this a projective map. These various concepts of projective map all amount to essentially the same thing because each has the same origin in terms of classes of vector space maps. For general vector space maps, just as for the isomorphisms, these classes are the proportional classes as the following exercise records.

Exercise 9.3 Let $f, g : \mathcal{V} \to \mathcal{W}$ be vector space maps from the finite-dimensional vector space \mathcal{V} and suppose that f and g induce the same projective map. Show that f and g are proportional by considering how they act on the basis vectors x_1, \ldots, x_k of a complementary subspace of their common kernel, and on $x_1 + \cdots + x_k$.

9.6 Central Projection

In V(V), fix a hyperplane t (the **target**), and fix a point C (the **center**) such that $C \notin t$. Given the point $P \in P(V) \setminus C$, we can form the line CP through C and P. CP will intersect t in a unique point Q. The assignment of $Q = CP \cap t$ to $P \in P(V) \setminus C$ is called **central projection** from the point C onto the hyperplane t. By also assigning the null flat of V(t) both to C and to the null flat of V(V), we get a function from $P^+(V)$ to $P^+(t)$ which turns out to be a projective map. Also, by restricting the domain to the points of

any hyperplane not containing C, and restricting the codomain to the points of t, we will see that a projective transformation between hyperplanes results, one that derives from the vector space self-map of projection onto t along C (Section 2.4).



Projecting P from C onto t

The figure above, not the usual schematic but rather a full depiction of the actual subspaces in \mathcal{V} , illustrates central projection in the simple case where \mathcal{V} is of dimension 3 and the target t is a projective line (a 2-dimensional subspace of \mathcal{V}). Let P be represented by the vector v. Then we can (uniquely) express v as v = x + w where $x \in C$ and $w \in t$ since $\mathcal{V} = C \oplus t$. But $w = v - x \in CP$, so $w \in CP \cap t$ and therefore w represents the point Q that we seek. Thus a vector representing Q may always be obtained by applying the projection onto t along C to any vector that represents P, thereby making the assignment of Q to P a projective map. C is the kernel of the projection onto t along C, so the points of any projective line s that does not contain S0 will be placed in one-to-one correspondence with the points of s1. Thus the

restriction of the projection onto t along C to a projective line s that does not contain C is a vector space isomorphism from s onto t, and the induced map from the points of s onto the points of t is a projective transformation, an example of a **perspectivity** between hyperplanes.

Although we have only illustrated here the case where \mathcal{V} has dimension 3, the same considerations clearly apply in any \mathcal{V} of finite dimension d. Also, as detailed in the following exercise, we may readily generalize to where center and target are any complementary subspaces in \mathcal{V} and obtain a concept of central projection from a center of any positive dimension n to a target of complementary dimension d-n.

Exercise 9.4 Let C be a subspace of dimension n > 0 in the vector space V of dimension d, and let T be a complement of C. Let P be a 1-dimensional subspace of V such that $P \cap C = 0$. Then the join CP of C and P intersects T in a 1-dimensional subspace Q. If v is a nonzero vector in P, then the projection of v onto T along C is a nonzero vector in Q.

9.7 Problems

1. Let f and g be isomorphisms between possibly infinite-dimensional vector spaces \mathcal{V} and \mathcal{W} , and let f and g be such that for all vectors v

$$g(v) = a(v) \cdot f(v)$$
 for a nonzero scalar $a(v)$.

Then for all vectors u and v, a(u) = a(v). Thus Theorem 99 holds in the infinite-dimensional case as well.

(Consider separately the cases where u and v represent the same point in P(V) and when they do not. Write g(u) two different ways in the former case and write g(u+v) in two different ways in the latter case.)

2. Suppose that \mathcal{V} is over the finite field of q elements. Then if $d \geq q$, d+1 distinct points of $\mathsf{P}(\mathcal{V})$, but no more, can be in general position. But if $2 \leq d < q$, then at least d+2 points of $\mathsf{P}(\mathcal{V})$ can be in general position.

10 Scalar Product Spaces

10.1 Pairings and Their Included Maps

Let \mathcal{V} and \mathcal{W} be vector spaces over the same field \mathcal{F} . We will refer to a bilinear functional $g: \mathcal{V} \times \mathcal{W} \to \mathcal{F}$ as a **pairing** of \mathcal{V} and \mathcal{W} . Consider the pairing $g: \mathcal{V} \times \mathcal{W} \to \mathcal{F}$. With the second argument of g held fixed at g, allowing the first argument to vary produces the values of a linear functional $g_{v} \in \mathcal{V}^{\top}$. Similarly, the values of a linear functional $g_{v} \in \mathcal{W}^{\top}$ are produced when the first argument is held fixed at v and the second argument is allowed to vary. Now, if we take g_{v} and let v vary, we get a map $g_{1}: \mathcal{V} \to \mathcal{W}^{\top}$, and if we take g_{w} and let v vary, we get a map v v. Thus v includes within itself the two maps v and v v which we will refer to as the **included** maps belonging to v. The various v are related by

$$g_{v}(w) = (g_1(v))(w) = g(v, w) = (g_2(w))(v) = g_{w}(v).$$

For any map $f: \mathcal{V} \to \mathcal{W}^{\top}$, (f(v))(w) is the value of a bilinear functional of v and w, and hence any such f is the g_1 included map of some pairing of \mathcal{V} and \mathcal{W} . By the same token, any map from \mathcal{W} to \mathcal{V}^{\top} can be a g_2 . However, each included map clearly determines the other, so only one of them may be specified arbitrarily.

10.2 Nondegeneracy

A pairing g is called **nondegenerate** if for each nonzero v, there is some w for which $g(v, w) \neq 0$, and for each nonzero w, there is some v for which $g(v, w) \neq 0$. A nondegenerate pairing $g : \mathcal{V} \times \mathcal{W} \to \mathcal{F}$ is sometimes referred to as a **perfect** (or **duality**) **pairing**, and is said to put \mathcal{V} in **duality** with \mathcal{W} .

Exercise 10.1 For any vector space \mathcal{V} over \mathcal{F} , the natural **evaluation pairing** $e: \mathcal{V}^{\top} \times \mathcal{V} \to \mathcal{F}$, defined by e(f, v) = f(v) for $f \in \mathcal{V}^{\top}$ and $v \in \mathcal{V}$, puts \mathcal{V}^{\top} in duality with \mathcal{V} . (Each $v \neq 0$ is part of some basis and therefore has a coordinate function v^{\top} .)

The nice thing about nondegeneracy is that it makes the included maps one-to-one, so they map different vectors to different functionals. For, if the distinct vectors s and t of \mathcal{V} both map to the same functional (so that

 $g_{t_} = g_{s_}$), then $g_{u_}$, where $u = t - s \neq 0$, is the zero functional on \mathcal{W} . Hence for the nonzero vector u, $g_{u_}(w) = g(u, w) = 0$ for all $w \in \mathcal{W}$, and g therefore fails to be nondegenerate.

Exercise 10.2 If the included maps are both one-to-one, the pairing is non-degenerate. (The one-to-one included maps send only the zero vector to the zero functional.)

If g is nondegenerate and one of the spaces is finite-dimensional, then so is the other. (\mathcal{W} is isomorphic to a subspace of \mathcal{V}^{\top} , so if \mathcal{V} is finite-dimensional, so is \mathcal{W} .) Supposing then that a nondegenerate g is a pairing of finite-dimensional spaces, we have $\dim \mathcal{W} \leq \dim \mathcal{V}^{\top} = \dim \mathcal{V}$ and also $\dim \mathcal{V} \leq \dim \mathcal{W}^{\top} = \dim \mathcal{W}$. Hence the spaces all have the same finite dimension and the included maps are therefore isomorphisms, since we know (Theorem 25) that a one-to-one map into an alias must also be onto.

The included maps belonging to a nondegenerate pairing g of a finite-dimensional \mathcal{V} with itself (which puts \mathcal{V} in duality with itself) are isomorphisms between \mathcal{V} and \mathcal{V}^{\top} , each of which allows any functional in \mathcal{V}^{\top} to be represented by a vector of \mathcal{V} in a basis-independent manner. Any $\varphi \in \mathcal{V}^{\top}$ is represented by $v_{\varphi} = g_1^{-1}(\varphi)$, and also by $w_{\varphi} = g_2^{-1}(\varphi)$, so that for any $w \in \mathcal{V}$

$$\varphi(w) = (g_1(g_1^{-1}(\varphi))(w) = g(g_1^{-1}(\varphi), w) = g(v_{\varphi}, w)$$

and for any $v \in \mathcal{V}$

$$g(v, w_{\varphi}) = g(v, g_2^{-1}(\varphi)) = (g_2(g_2^{-1}(\varphi))(v) = \varphi(v).$$

10.3 Orthogonality, Scalar Products

Designating an appropriate pairing g of \mathcal{V} with itself will give us the means to define orthogonality on a vector space by saying that the vectors u and v are **orthogonal** (written $u \perp v$) if g(u,v) = 0. Requiring nondegeneracy will insure that no nonzero vector is orthogonal to all nonzero vectors. However, we also want to rule out the undesirable possibility that g(u,v) = 0 while $g(v,u) \neq 0$. Thus we will also insist that g be **reflexive**: g(u,v) = 0 if and only if g(v,u) = 0. g will be reflexive if it is symmetric (g(u,v) = g(v,u)) for all u,v, or if it is alternating (g(v,v) = 0) for all v, which implies g(u,v) = -g(v,u) for all u,v. Here we will be confining our attention to the case where g is symmetric and nondegenerate. By designating a specific

symmetric nondegenerate self-pairing as the **scalar product** on a vector space, the vector space with its designated scalar product becomes a **scalar product space**. The scalar product $g: \mathcal{V} \times \mathcal{V} \to \mathcal{F}$ then provides the orthogonality relation \bot on \mathcal{V} .

Exercise 10.3 A symmetric pairing g of a vector space with itself is nondegenerate if and only if g(v, w) = 0 for all w implies v = 0.

Notice that the two included maps of a scalar product are identical, and abusing notation somewhat, we will use the same letter g to refer both to the included map and to the scalar product itself. $g: \mathcal{V} \times \mathcal{V} \to \mathcal{F}$ and $g: \mathcal{V} \to \mathcal{V}^{\top}$ will not be readily confused. When \mathcal{V} is finite-dimensional, the scalar product's included map g is an isomorphism that allows each $\varphi \in \mathcal{V}^{\top}$ to be represented by the vector $g^{-1}(\varphi)$ via

$$\varphi(w) = g(g^{-1}(\varphi), w),$$

so that the evaluation of the functional φ at any vector w can be replaced by the evaluation of the scalar product at $(g^{-1}(\varphi), w)$. In slightly different notation, with e being the evaluation pairing introduced in Exercise 10.1 above, the formula reads

$$e(\varphi, w) = g(g^{-1}(\varphi), w),$$

which shows how the two seemingly different pairings, e and g, are essentially the same when \mathcal{V} is a finite-dimensional scalar product space. The natural nondegenerate pairing e between \mathcal{V}^{\top} and \mathcal{V} thus takes on various interpretations on \mathcal{V} depending on the choice of scalar product on the finite-dimensional space. For instance, given a particular vector w, the functionals φ such that $\varphi(w) = 0$ determine by $v = g^{-1}(\varphi)$ the vectors v for which $v \perp w$. Another example is the interpretation of the dual of a basis for \mathcal{V} as another basis for \mathcal{V} .

10.4 Reciprocal Basis

Notice: We assume for the remainder of the chapter that we are treating a scalar product space \mathcal{V} of finite dimension d over the field \mathcal{F} , and that \mathcal{V} has the scalar product g.

Let $\mathcal{B} = \{x_1, \dots, x_d\}$ be a basis for \mathcal{V} and $\mathcal{B}^{\top} = \{x_1^{\top}, \dots, x_d^{\top}\}$ its dual on \mathcal{V}^{\top} . For each i, let $x_i^{\perp} = g^{-1}(x_i^{\top})$. Then $\mathcal{B}^{\perp} = \{x_1^{\perp}, \dots, x_d^{\perp}\}$ is another basis for \mathcal{V} , known as the **reciprocal** of \mathcal{B} . Acting through the scalar product, the elements of \mathcal{B}^{\perp} behave the same as the coordinate functions from which they derive via g^{-1} and therefore satisfy the **biorthogonality** conditions

$$g(x_i^{\perp}, x_j) = x_i^{\top}(x_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for all i and j. On the other hand, if some basis $\{y_1, \ldots, y_d\}$ satisfies $g(y_i, x_j) = (g(y_i))(x_j) = \delta_{i,j}$ for all i and j, then $g(y_i)$ must by definition be the coordinate function x_i^{\top} , which then makes $y_i = g^{-1}(x_i^{\top}) = x_i^{\perp}$. We therefore conclude that biorthogonality characterizes the reciprocal. Hence, if we replace \mathcal{B} with \mathcal{B}^{\perp} , the original biorthogonality formula displayed above tells us that $(\mathcal{B}^{\perp})^{\perp} = \mathcal{B}$.

For each element x_j of the above basis \mathcal{B} , $g(x_j) \in \mathcal{V}^{\top}$ has a (unique) dual basis representation of the form

$$g(x_j) = \sum_{i=1}^d g_{i,j} \cdot x_i^{\top}.$$

Applying this representation to x_k yields $g_{k,j} = g(x_j, x_k)$, so we have the explicit formula

$$g(x_j) = \sum_{i=1}^d g(x_i, x_j) \cdot x_i^{\top}.$$

Applying g^{-1} to this result, we get the following formula that expresses each vector of \mathcal{B} in terms of the vectors of \mathcal{B}^{\perp} as

$$x_j = \sum_{i=1}^d g(x_i, x_j) \cdot x_i^{\perp}.$$

Each element of \mathcal{B}^{\perp} thus is given in terms of the elements of \mathcal{B} by

$$x_j^{\perp} = \sum_{i=1}^d g^{i,j} \cdot x_i,$$

where $g^{i,j}$ is the element in the *i*th row and *j*th column of the *inverse* of

$$[g(x_i, x_j)] = [g_{i,j}] = \begin{bmatrix} g(x_1, x_1) & \cdots & g(x_1, x_d) \\ \vdots & \cdots & \vdots \\ g(x_d, x_1) & \cdots & g(x_d, x_d) \end{bmatrix}.$$

This we may verify by noting that since the $g^{i,j}$ satisfy

$$\sum_{k=1}^{d} g_{i,k} g^{k,j} = \delta_i^j = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases}$$

we have

$$\sum_{i=1}^{d} g^{i,j} \cdot x_i = \sum_{i=1}^{d} g^{i,j} \sum_{k=1} g_{k,i} \cdot x_k^{\perp} = \sum_{k=1}^{d} \sum_{i=1}^{d} g_{k,i} g^{i,j} \cdot x_k^{\perp} = \sum_{k=1}^{d} \delta_k^j \cdot x_k^{\perp} = x_j^{\perp}.$$

Exercise 10.4 $x_1 \wedge \cdots \wedge x_d = G \cdot x_1^{\perp} \wedge \cdots \wedge x_d^{\perp}$ where $G = \det[g(x_i, x_j)]$.

10.5 Related Scalar Product for Dual Space

Given the scalar product g on \mathcal{V} , we define the **related** scalar product \widetilde{g} on \mathcal{V}^{\top} . This we do in a natural way by mapping the elements of \mathcal{V}^{\top} back to \mathcal{V} via g^{-1} , and then declaring the scalar product of the mapped elements to be the scalar product of the elements themselves. Thus for $\varphi, \psi \in \mathcal{V}^{\top}$, we define $\widetilde{g}: \mathcal{V}^{\top} \times \mathcal{V}^{\top} \to \mathcal{F}$ by $\widetilde{g}(\varphi, \psi) = g(g^{-1}(\varphi), g^{-1}(\psi))$. It is easy to see that this \widetilde{g} is indeed a scalar product. Note that $g(v, w) = \widetilde{g}(g(v), g(w))$, so the included map $g: \mathcal{V} \to \mathcal{V}^{\top}$ is more than just an isomorphism – it also preserves scalar product. We call such a map an **isometry**. It makes \mathcal{V} and \mathcal{V}^{\top} into **isometric** scalar product spaces which are then scalar product space aliases, and we have justification to view them as two versions of the same scalar product space by identifying each $v \in \mathcal{V}$ with $g(v) \in \mathcal{V}^{\top}$.

For the elements of the dual basis $\mathcal{B}^{\top} = \{x_1^{\top}, \dots, x_d^{\top}\}$ of the basis $\mathcal{B} = \{x_1, \dots, x_d\}$ for \mathcal{V} , we then have

$$\widetilde{g}(x_1^\top, x_j^\top) = g\left(g^{-1}(x_i^\top), g^{-1}(x_j^\top)\right) = g\left(x_i^\perp, x_j^\perp\right),$$

and using the result we found in the previous section, the new included map $\tilde{g}: \mathcal{V}^{\top} \to \mathcal{V}$ has the (unique) basis representation

$$\widetilde{g}(x_j^{\top}) = \sum_{i=1}^d g\left(x_i^{\perp}, x_j^{\perp}\right) \cdot x_i.$$

Swapping \mathcal{B} and \mathcal{B}^{\perp} , a formula derived in the previous section becomes the equally valid formula

$$x_j^{\perp} = \sum_{i=1}^d g(x_i^{\perp}, x_j^{\perp}) \cdot x_i$$

which has the same right hand side as the preceding basis representation formula, so we see that $\tilde{g}(x_j^{\top}) = x_j^{\perp}$. The included map \tilde{g} must therefore be g^{-1} . Applying $g = \tilde{g}^{-1}$ to both sides of the basis representation formula for $\tilde{g}(x_i^{\top})$, we get

$$x_j^{\top} = \sum_{i=1}^d g\left(x_i^{\perp}, x_j^{\perp}\right) \cdot g(x_i),$$

or

$$x_j^{\top} = \sum_{i=1}^d g\left(x_i^{\perp}, x_j^{\perp}\right) \cdot \left(x_i^{\top}\right)^{\perp}.$$

As we already know from the previous section, the inverse relationship is

$$(x_i^{\top})^{\perp} = \widetilde{g}^{-1}(x_j) = g(x_j) = \sum_{i=1}^d g(x_i, x_j) \cdot x_i^{\top}.$$

Exercise 10.5 $\left[g\left(x_{i}^{\perp}, x_{j}^{\perp}\right)\right] = \left[g^{i,j}\right], i. e., \left[g\left(x_{i}^{\perp}, x_{j}^{\perp}\right)\right] = \left[g\left(x_{i}, x_{j}\right)\right]^{-1} = \left[g_{i,j}\right]^{-1}.$

10.6 Some Notational Conventions

Notational conventions abound in practice, and we note some of them now. It is not unreasonable to view \tilde{g} as extending g to \mathcal{V}^{\top} , and it would not be unreasonable to just write g for both (remembering that the included map of the g on \mathcal{V}^{\top} is the inverse of the included map of the original g on \mathcal{V}). Unless explicitly stated otherwise, we will assume henceforth that the related scalar product, no tilde required, is being used on \mathcal{V}^{\top} . In the literature, either scalar product applied to a pair of vector or functional arguments is often seen written (v, w) or (φ, ψ) , entirely omitting g at the front, but we will avoid this practice here.

Relative to a fixed basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ for \mathcal{V} and its dual for \mathcal{V}^{\top} , the matrix $[g(x_i, x_j)]$ of the included map of the original g may be abbreviated as $[g_{ij}]$ (without any comma between the separate subscripts i and j), and the matrix $[g(x_i, x_j)]^{-1}$ of the related scalar product on \mathcal{V}^{\top} may similarly be abbreviated as $[g^{ij}]$. The typical element g_{ij} of the matrix $[g_{ij}]$ is often used to designate the g on \mathcal{V} , and similarly g^{ij} to designate the related g on \mathcal{V}^{\top} . By the same token, v^i designates the vector $v = v^1 \cdot x_1 + \cdots + v^d \cdot x_d$ and φ_i designates the functional $\varphi = \varphi_1 \cdot x_1^{\top} + \cdots + \varphi_d \cdot x_d^{\top}$. The detailed evaluation of

g(v, w) is usually then written as $g_{ij}v^iw^j$ which, since i appears twice (only), as does j, is taken to mean that summation over both i and j has tacitly been performed over their known ranges of 1 to d. This convention, called the summation convention (attributed to Albert Einstein), can save a lot of ink by omitting many \sum signs. Similarly, the detailed evaluation of the scalar product $g(\varphi, \psi) = \tilde{g}(\varphi, \psi)$ of the two functionals φ and ψ appears as $g^{ij}\varphi_i\psi_j$. Other examples are the evaluation of $\varphi = g(v)$ as $v_i = g_{ij}v^j$ (the vector v is converted to a functional φ usually designated v_i rather than φ_i) and the reverse $v = g^{-1}(\varphi)$ as $v^i = g^{ij}v_j$. Note that $v_iw^i = g_{ij}v^jw^i = g(v)(w) = g(v, w)$ and $g^{ij}g_{jk}v^k = \delta^i_kv^k = v^i$.

This component-style notation, including the summation convention, is usually referred to as *tensor notation*. It has been a very popular notation in the past, and is still widely used today in those applications where it has gained a traditional foothold. Some concepts are easier to express using it, but it can also bury concepts in forests of subscripts and superscripts. The above samples of tensor notation used in scalar product calculations were included to show how scalar product concepts are often handled notationally in the literature, and not because we are going to adopt this notation here.

10.7 Standard Scalar Products

We will now define standard scalar products on the finite-dimensional vector space \mathcal{V} over the field \mathcal{F} . As we know, we may determine a bilinear function by giving its value on each pair (x_i, x_i) of basis vectors of whatever basis $\{x_1,\ldots,x_d\}$ we might choose. The bilinear functional $g:\mathcal{V}\times\mathcal{V}\to\mathcal{F}$ will be called a standard scalar product if for the chosen basis $\{x_1, \ldots, x_d\}$ we have $g(x_i, x_j) = 0$ whenever $i \neq j$ and each $g(x_i, x_i) = \eta_i$ is either +1or -1. A standard scalar product will be called **definite** if the η_i are all equal, and any other standard scalar product will be called **indefinite**. The standard scalar product for which $\eta_i = +1$ for all i will be referred to as the positive standard scalar product or as the usual standard scalar product. With the standard basis $\{(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$ as the chosen basis, the usual standard scalar product on the real d-dimensional space \mathbb{R}^d is the well-known Euclidean inner product, or dot product. A standard scalar product with chosen basis $\{x_1,\ldots,x_d\}$ always makes $x_i \perp x_j$ whenever $i \neq j$, so the chosen basis is always an **orthogonal basis** under a standard scalar product, and because the scalar product of a chosen basis vector with itself is ± 1 , the chosen basis for a standard scalar product is more specifically an

orthonormal basis.

Exercise 10.6 A standard scalar product is indeed a scalar product. (For the nondegeneracy, Exercise 10.3 above may be applied using each vector from the chosen basis as a w.)

Exercise 10.7 On real 2-dimensional space \mathbb{R}^2 , there is a standard scalar product that makes some nonzero vector orthogonal to itself. However, \mathbb{R}^d with the Euclidean inner product as its scalar product has no nonzero vector that is orthogonal to itself.

With a standard scalar product, the vectors of the reciprocal of the chosen basis can be found in terms of those of the chosen basis without computation.

Exercise 10.8 The included map g for a standard scalar product satisfies $g(x_i) = \eta_i \cdot x_i^{\top}$ for each basis vector x_i of the chosen basis, and also then $x_i^{\perp} = g^{-1}(x_i^{\top}) = \eta_i \cdot x_i$ for each i.

A scalar product that makes a particular chosen basis orthonormal can always be imposed on a vector space, of course. However, it is not necessarily true that an orthonormal basis exists for each possible scalar product that can be specified on a vector space. As long as $1+1 \neq 0$ in \mathcal{F} , an orthogonal basis can always be found, but due to a lack of square roots in \mathcal{F} , it may be impossible to find any orthogonal basis that can be normalized. However, over the real numbers, normalization is always possible, and in fact, over the reals, the number of $g(x_i, x_i)$ that equal -1 is always the same for every orthonormal basis of a given scalar product space due to the well-known Sylvester's Law of Inertia (named for James Joseph Sylvester who published a proof in 1852).

Exercise 10.9 Let $\mathcal{F} = \{0, 1\}$ and let $\mathcal{V} = \mathcal{F}^2$. Let x_1 and x_2 be the standard basis vectors and let $[g(x_i, x_j] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then for this g, \mathcal{V} has no orthogonal basis.

10.8 Orthogonal Subspaces and Hodge Star

Corresponding to each subspace $\mathcal{X} \triangleleft \mathcal{V}$ is its **orthogonal subspace** \mathcal{X}^{\perp} consisting of all the vectors from \mathcal{V} that are orthogonal to every vector of \mathcal{X} .

A vector v is thus in \mathcal{X}^{\perp} if and only if g(v,x) = g(v)(x) = 0 for all $x \in \mathcal{X}$ or hence if and only if g(v) is in the annihilator of \mathcal{X} (Section 3.3). Thus $\mathcal{X}^{\perp} = g^{-1}(\mathcal{X}^0)$, where $\mathcal{X}^0 \triangleleft \mathcal{V}^{\top}$ is the annihilator of $\mathcal{X} \triangleleft \mathcal{V}$, and \mathcal{X}^{\perp} is therefore just \mathcal{X}^0 being interpreted in \mathcal{V} by using the scalar product's ability to represent linear functionals as vectors.

Suppose now that we fix a basis $\mathcal{B}_H = \{x_1, \dots, x_d\}$ and use it, as we did in Theorem 90, in defining the map $H : \bigwedge \mathcal{V} \to \bigwedge \mathcal{V}^{\top}$ that maps each blade to a corresponding annihilator blade. Then replacing each x_i^{\top} in the corresponding annihilator blade in $\bigwedge \mathcal{V}^{\top}$ by x_i^{\perp} changes it into a corresponding **orthogonal blade** in $\bigwedge \mathcal{V}$. The process of replacing each x_i^{\top} by x_i^{\perp} may be viewed as extending g^{-1} to $\bigwedge \mathcal{V}^{\top}$ and the resulting isomorphism, (which, incidentally, is easily seen to be independent of the basis choice) is denoted $\bigwedge g^{-1} : \bigwedge \mathcal{V}^{\top} \to \bigwedge \mathcal{V}$. The composite $\bigwedge g^{-1} \circ H$ will be denoted by $*: \bigwedge \mathcal{V} \to \bigwedge \mathcal{V}$ and will be called the **Hodge star**, for Scottish geometer William Vallance Douglas Hodge (1903 – 1975), although it is essentially the same as an operator that Grassmann used. Thus the "annihilator blade map" H is reinterpreted as the "orthogonal blade map" * for blades and the subspaces they represent. Applying * to exterior products of elements of \mathcal{B}_H gives

$$*(x_{i_1} \wedge \cdots \wedge x_{i_n}) = (-1)^{\rho} \cdot x_{i_{n+1}}^{\perp} \wedge \cdots \wedge x_{i_d}^{\perp},$$

where ρ is the permutation i_1, \ldots, i_d of $1, \ldots, d$ and $(-1)^{\rho} = +1$ or -1 according as ρ is even or odd. Employing the usual standard scalar product with \mathcal{B}_H as its chosen basis, we then have, for example, $*(x_1 \wedge \cdots \wedge x_n) = x_{n+1} \wedge \cdots \wedge x_d$.

Exercise 10.10 In \mathbb{R}^2 , with \mathcal{B}_H the standard basis $x_1 = (1,0)$, $x_2 = (0,1)$, compute *((a,b)) and check the orthogonality for each of these scalar products g with matrix $\begin{bmatrix} g(x_1,x_1) & g(x_1,x_2) \\ g(x_2,x_1) & g(x_2,x_2) \end{bmatrix} =$

a)
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
, b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, d) $\begin{bmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix}$.

Exercise 10.11 For $\mathcal{X} \triangleleft \mathcal{V}$, dim $\mathcal{X}^{\perp} = \dim \mathcal{V} - \dim \mathcal{X}$.

Exercise 10.12 For $\mathcal{X} \triangleleft \mathcal{V}$, $\mathcal{V} = \mathcal{X} \oplus \mathcal{X}^{\perp}$ if and only if no nonzero vector of \mathcal{X} is orthogonal to itself.

Exercise 10.13 For $\mathcal{X} \triangleleft \mathcal{V}$, $\mathcal{X}^{\perp \perp} = (\mathcal{X}^{\perp})^{\perp} = \mathcal{X}$ since $\mathcal{X} \triangleleft \mathcal{X}^{\perp \perp}$ and $\dim \mathcal{X}^{\perp \perp} = \dim \mathcal{X}$.

The Hodge star was defined above by the simple formula $\bigwedge g^{-1} \circ H$. This formula could additionally be scaled, as some authors do, in order to meet some particular normalization criterion, such as making a blade and its star in some sense represent the same geometric content. For example, $\bigwedge g^{-1} \circ H$ is sometimes scaled up by the factor $\sqrt{|\det[g(x_i, x_j)]|}$ when \mathcal{F} is the field of real numbers. However, for the time being at least, \mathcal{F} will be kept general, and the unscaled simple formula will continue to be our definition.

10.9 Scalar Product on Exterior Powers

We will now show that the scalar product g on \mathcal{V} may be extended to each $\bigwedge^p \mathcal{V}$ as the bilinear function g that on p-blades has the value

$$g(v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p) = \det [g(v_i, w_i)].$$

Thus we need to show that there is a unique bilinear function $g: \bigwedge^p \mathcal{V} \times \bigwedge^p \mathcal{V} \to \mathcal{F}$ that satisfies the above specification, and that this g is symmetric and nondegenerate. (For p=0, by convention we put g(1,1)=1 so that $g(a\cdot 1,b\cdot 1)=ab$. The cases p=0 or 1 need no further attention, so we assume p>1 in the following treatment.)

For fixed (v_1, \ldots, v_p) , det $[g(v_i, w_j)]$ is an alternating p-linear function of (w_1, \ldots, w_p) and by Theorem 66 there is a unique $g_{v_1, \ldots, v_p} \in (\bigwedge^p \mathcal{V})^\top$ such that $g_{v_1, \ldots, v_p}(w_1 \wedge \cdots \wedge w_p) = \det [g(v_i, w_j)]$. Now the assignment of g_{v_1, \ldots, v_p} to (v_1, \ldots, v_p) is an alternating p-linear function and hence there is a unique map $g_{\wedge} : \bigwedge^p \mathcal{V} \to (\bigwedge^p \mathcal{V})^\top$ such that $g_{\wedge}(v_1 \wedge \cdots \wedge v_p) = g_{v_1, \ldots, v_p}$ and therefore such that $g_{\wedge}(v_1 \wedge \cdots \wedge v_p)(w_1 \wedge \cdots \wedge w_p) = \det [g(v_i, w_j)]$. Evidently this g_{\wedge} is an included map of the uniquely determined bilinear functional $g : \bigwedge^p \mathcal{V} \times \bigwedge^p \mathcal{V} \to \mathcal{F}$ given by $g(s,t) = g_{\wedge}(s)(t)$ for any elements s,t of $\bigwedge^p \mathcal{V}$.

The symmetry of the g we have just defined on all of $\bigwedge^p \mathcal{V}$ follows readily from the symmetry of g on \mathcal{V} , and the equality of the determinant of a matrix with that of its transpose.

We will show nondegeneracy by showing that the included map is one-to-one. Let \mathcal{V} have the basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ from which we make a basis $\mathcal{B}^{\wedge p}$ for $\bigwedge^p \mathcal{V}$ in the usual manner by taking exterior products of the elements

of each subset of p elements of \mathcal{B} . Elements of \mathcal{B}^{\wedge^p} will denoted $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, $x_J = x_{j_1} \wedge \cdots \wedge x_{j_p}$, etc. using capital-letter subscripts that denote the multi-index subscript sets $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_p\}$ of the p elements of \mathcal{B} that are involved. Indexing now with these multi-indices, our familiar explicit dual basis representation formula for the effect of the included map (now also called g) on basis vectors from \mathcal{B}^{\wedge^p} appears as

$$g(x_J) = \sum_{|I|=p} g(x_I, x_J) \cdot (x_I)^\top,$$

with the scalar product on basis vector pairs being

$$g(x_I, x_J) = g(x_{i_1} \wedge \cdots \wedge x_{i_p}, x_{j_1} \wedge \cdots \wedge x_{j_p}) = \det \left[g(x_{i_k}, x_{j_l}) \right]$$

of course.

The concept of the pth exterior power of a map, introduced by Exercise 5.11, will now find employment. Consider the pth exterior power $\bigwedge^p g$: $\bigwedge^p \mathcal{V} \to \bigwedge^p \mathcal{V}^\top$ of the included map g of the original scalar product on \mathcal{V} . We have

$$\bigwedge^p g(x_J) = g(x_{j_1}) \wedge \cdots \wedge g(x_{j_p}) = \left(\sum_{i_1=1}^d g(x_{i_1}, x_{j_1}) \cdot x_{i_1}^\top\right) \wedge \cdots \wedge \left(\sum_{i_p=1}^d g(x_{i_p}, x_{j_p}) \cdot x_{i_p}^\top\right)$$

from which it follows that

$$\bigwedge^p g(x_J) = \sum_{|I|=p} \det \left[g(x_{i_k}, x_{j_l}) \right] \cdot x_I^{\top}.$$

So we see that the new included map has a basis representation

$$g(x_J) = \sum_{|I|=p} g_{I,J} \cdot (x_I)^\top,$$

and the pth exterior power of the original included map has a basis representation

$$\bigwedge^p g(x_J) = \sum_{|I|=p} g_{I,J} \cdot x_I^\top,$$

both with the same coefficients $g_{I,J}$, namely

$$g_{I,J} = \det \left[g(x_{i_k}, x_{j_l}) \right].$$

Therefore, applying Proposition 70, if one of these maps is invertible, then so is the other one. Now, the invertibility of the original included map $g: \mathcal{V} \to \mathcal{V}^{\top}$ makes its pth exterior power invertible as well, because the $g(x_i)$ being a basis for \mathcal{V}^{\top} make the $\bigwedge^p g(x_I)$ a basis for $\bigwedge^p \mathcal{V}^{\top}$. Therefore the new included map is also invertible and our new symmetric bilinear g is indeed nondegenerate as we wished to show. What is going on here is rather transparent: the pth exterior power of the included map $g: \mathcal{V} \to \mathcal{V}^{\top}$ followed by the isomorphism: $\widehat{\Phi}: \bigwedge^p \mathcal{V}^{\top} \to (\bigwedge^p \mathcal{V})^{\top}$ of Theorem 71, evidently is the included map of the extension. That is, we have the formula $g(s,t) = ((\widehat{\Phi} \circ \bigwedge^p g)(s))(t)$, valid for all $(s,t) \in \bigwedge^p \mathcal{V} \times \bigwedge^p \mathcal{V}$. If we agree to identify $(\bigwedge^p \mathcal{V})^{\top}$ with $\bigwedge^p \mathcal{V}^{\top}$ through the isomorphism $\widehat{\Phi}$, then $\bigwedge^p g$ "is" the included map of our scalar product extension to $\bigwedge^p \mathcal{V}$.

The Hodge star provides an isomorphic mapping of $\bigwedge^p \mathcal{V}$ with $\bigwedge^{d-p} \mathcal{V}$ that is also an isometry (except for scale), as the following proposition records.

Proposition 101 Let $s, r \in \bigwedge^p \mathcal{V}$, let $\{x_1, \ldots, x_d\}$ be the basis for \mathcal{V} used in defining *, and let $G = \det[g(x_i, x_i)]$. Then

$$g(r,s) = Gg(*r,*s).$$

Proof: It suffices to prove the result for $r = x_I$ and $s = x_J$. Now

$$g(x_I, x_J) = \det \begin{bmatrix} g_{i_1, j_1} & \cdots & g_{i_1, j_p} \\ \vdots & \cdots & \vdots \\ g_{i_p, j_1} & \cdots & g_{i_p, j_p} \end{bmatrix}$$

and in light of Exercise 10.5

$$g(*x_I, *x_J) = (-1)^{\rho} (-1)^{\sigma} \det \begin{bmatrix} g^{i_{p+1}, j_{p+1}} & \cdots & g^{i_{p+1}, j_d} \\ \vdots & \cdots & \vdots \\ g^{i_d, j_{p+1}} & \cdots & g^{i_d, j_d} \end{bmatrix}$$

where ρ and σ are the respective permutations i_1, \ldots, i_d and j_1, \ldots, j_d of $\{1, \ldots, d\}$. By Jacobi's Determinant Identity (Lemma 89)

$$\det \begin{bmatrix} g_{i_1,j_1} & \cdots & g_{i_1,j_p} \\ \vdots & \cdots & \vdots \\ g_{i_p,j_1} & \cdots & g_{i_p,j_p} \end{bmatrix} = \det \begin{bmatrix} g_{i_1,j_1} & \cdots & g_{i_1,j_d} \\ \vdots & \cdots & \vdots \\ g_{i_d,j_1} & \cdots & g_{i_d,j_d} \end{bmatrix} \det \begin{bmatrix} g^{i_{p+1},j_{p+1}} & \cdots & g^{i_{p+1},j_d} \\ \vdots & \cdots & \vdots \\ g^{i_d,j_{p+1}} & \cdots & g^{i_d,j_d} \end{bmatrix}$$

from which it follows that $g(x_I, x_J) = Gg(*x_I, *x_J)$ once we have attributed the $(-1)^{\rho} (-1)^{\sigma}$ to the rearranged rows and columns of the big determinant.

10.10 Another Way to Define *

A version of the Hodge star may be defined using the scalar products defined on the $\bigwedge^p \mathcal{V}$ in the previous section. Let $\mathcal{B} = \{x_1, \dots, x_d\}$ be a designated basis for \mathcal{V} and consider the exterior product $s \wedge t$ where s is a fixed element of $\bigwedge^p \mathcal{V}$ and t is allowed to vary over $\bigwedge^{d-p} \mathcal{V}$. Each such $s \wedge t$ is of the form $c_s(t) \cdot x_1 \wedge \cdots \wedge x_d$ where $c_s \in \left(\bigwedge^{d-p} \mathcal{V}\right)^{\top}$. Hence there is a unique element $*s \in \bigwedge^{d-p} \mathcal{V}$ such that $g(*s,t) = c_s(t) = (x_1 \wedge \cdots \wedge x_d)^{\top} (s \wedge t)$ and this unique element *s for which

$$s \wedge t = g(*s, t) \cdot x_1 \wedge \cdots \wedge x_d$$

will be the new version of the Hodge star on $\bigwedge^p \mathcal{V}$ that we will now scrutinize and compare with the original version. We will focus on elements s of degree p (meaning those in $\bigwedge^p \mathcal{V}$), but we do intend this * to be applicable to arbitrary elements of $\bigwedge \mathcal{V}$ by applying it separately to the components of each degree and summing the results.

If we have an element $r \in \bigwedge^{d-p} \mathcal{V}$ that we allege is our new *s for some $s \in \bigwedge^p \mathcal{V}$, we can verify that by showing that g(r,t) is the coefficient of $x_1 \wedge \cdots \wedge x_d$ in $s \wedge t$ for all t in a basis for $\bigwedge^{d-p} \mathcal{V}$. If we show this for such an r corresponding to each s that is in a basis for $\bigwedge^p \mathcal{V}$, then we will have completely determined the new * on $\bigwedge^p \mathcal{V}$.

Let us see what happens when we choose the designated basis \mathcal{B} to be the basis \mathcal{B}_H used in defining the annihilator blade map H, with exactly the same assignment of subscript labels to the basis vectors x_i . With s and t equal to the respective basis monomials $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$ and $x_J = x_{j_{p+1}} \wedge \cdots \wedge x_{j_d}$, we have

$$s \wedge t = x_I \wedge x_J = \varepsilon_{I,J} \cdot x_1 \wedge \cdots \wedge x_d$$

where $\varepsilon_{I,J}=0$ if $\{j_{p+1},\ldots,j_d\}$ is not complementary to $\{i_1,\ldots,i_p\}$ in $\{1,\ldots,d\}$, and otherwise $\varepsilon_{I,J}=(-1)^\sigma$ with σ the permutation $i_1,\ldots,i_p,j_{p+1},\ldots,j_d$ of $1,\ldots,d$. The original Hodge star gives

$$*x_I = *(x_{i_1} \wedge \cdots \wedge x_{i_p}) = (-1)^{\rho} \cdot x_{i_{p+1}}^{\perp} \wedge \cdots \wedge x_{i_d}^{\perp}$$

where ρ is the permutation i_1, \ldots, i_d of $1, \ldots, d$, so for this original $*x_I$,

$$g(*x_{I}, x_{J}) = (-1)^{\rho} \det \begin{bmatrix} g(x_{i_{p+1}}^{\perp}, x_{j_{p+1}}) & \cdots & g(x_{i_{p+1}}^{\perp}, x_{j_{d}}) \\ \vdots & \cdots & \vdots \\ g(x_{i_{d}}^{\perp}, x_{j_{p+1}}) & \cdots & g(x_{i_{d}}^{\perp}, x_{j_{d}}) \end{bmatrix}$$
$$= (-1)^{\rho} \det \begin{bmatrix} x_{i_{p+1}}^{\top}(x_{j_{p+1}}) & \cdots & x_{i_{p+1}}^{\top}(x_{j_{d}}) \\ \vdots & \cdots & \vdots \\ x_{i_{d}}^{\top}(x_{j_{p+1}}) & \cdots & x_{i_{d}}^{\top}(x_{j_{d}}) \end{bmatrix}.$$

If J is not complementary to I in $\{1,\ldots,d\}$ then j_{p+1},\ldots,j_d is not a permutation of i_{p+1},\ldots,i_d and some j_{p+k} equals none of i_{p+1},\ldots,i_d so that the kth column is all zeroes and the latter determinant vanishes. On the other hand, if J is complementary to I in $\{1,\ldots,d\}$ then $J=\{i_{p+1},\ldots,i_d\}$ and there is no reason why we cannot assume that $j_{p+1}=i_{p+1},\ldots,j_d=i_d$, which then makes the determinant equal to 1 and makes σ equal to ρ . Hence using the same basis ordered the same way, our new version of the Hodge star is exactly the same as the original version.

Exercise 10.14 For a given scalar product space, using $\mathcal{B}' = \{x'_1, \ldots, x'_d\}$ instead of $\mathcal{B} = \{x_1, \ldots, x_d\}$ to define the Hodge star gives $h \cdot *$ instead of *, where h is the nonzero factor, independent of p, such that $h \cdot x'_1 \wedge \cdots \wedge x'_d = x_1 \wedge \cdots \wedge x_d$. Thus, no matter which definition is used for either, or what basis is used in defining either, for any two of our Hodge stars, over all of $\wedge \mathcal{V}$ the values of one are the same constant scalar multiple of the values of the other. That is, ignoring scale, for a given scalar product space \mathcal{V} all of our Hodge stars on $\wedge \mathcal{V}$ are identical, and the scale depends only on the basis choice (including labeling) for \mathcal{V} used in each definition.

Exercise 10.15 (Continuation) Suppose that for the scalar product g, the bases \mathcal{B} and \mathcal{B}' have the same **Gram determinant**, i.e., $\det [g(x_i, x_j)] = \det [g(x_i', x_j')]$, or equivalently, $g(x_1 \wedge \cdots \wedge x_d, x_1 \wedge \cdots \wedge x_d) = g(x_1' \wedge \cdots \wedge x_d', x_1' \wedge \cdots \wedge x_d')$. Taking it as a given that in a field, 1 has only itself and -1 as square roots, \mathcal{B} and \mathcal{B}' then produce the same Hodge star up to sign.

The following result is now apparent.

Proposition 102 Two bases of a given scalar product space yield the same Hodge star up to sign if and only if they have the same Gram determinant.

Finally, we obtain some more formulas of interest, and some conclusions based on them. Putting t = *r in the defining formula above gives

$$s \wedge *r = q (*s, *r) \cdot x_1 \wedge \cdots \wedge x_d$$

Similarly, we find

$$r \wedge *s = g(*r, *s) \cdot x_1 \wedge \cdots \wedge x_d = g(*s, *r) \cdot x_1 \wedge \cdots \wedge x_d,$$

and therefore for all $r, s \in \bigwedge^p \mathcal{V}$ we have

$$r \wedge *s = s \wedge *r$$
.

Applying Proposition 101, we also get

$$s \wedge *r = G^{-1}g(r,s) \cdot x_1 \wedge \cdots \wedge x_d,$$

where $G = \det[g(x_i, x_j)]$. The results of the next two exercises then follow readily.

Exercise 10.16 For all $r, s \in \bigwedge^p \mathcal{V}$,

$$g(r,s) = G \cdot * (r \wedge *s) = G \cdot * (s \wedge *r).$$

Exercise 10.17 For any $r \in \bigwedge^p \mathcal{V}$,

$$*r = G^{-1} \sum_{|I|=p} (-1)^{\rho} g(r, x_I) \cdot x_{\overline{I}},$$

where $I = \{i_1, ..., i_p\}, \overline{I} = \{i_{p+1}, ..., i_d\}, I \cup \overline{I} = \{1, ..., d\}, and$

$$\rho = \left(\begin{array}{ccc} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{array}\right).$$

Thus, not only do we have a nice expansion formula for *r, but we may also conclude that *r has yet another definition as the unique $t \in \bigwedge^{d-p} \mathcal{V}$ for which $s \wedge t = G^{-1}g(r, s) \cdot x_1 \wedge \cdots \wedge x_d$ for all $s \in \bigwedge^p \mathcal{V}$.

10.11 Hodge Stars for the Dual Space

We will now define a related Hodge star $\widetilde{*}$ on $\bigwedge \mathcal{V}^{\top}$ in a way similar to how we defined \widetilde{g} on \mathcal{V}^{\top} . We will use $\bigwedge g^{-1}$ to map a blade from $\bigwedge \mathcal{V}^{\top}$ to $\bigwedge \mathcal{V}$, use * to get an orthogonal blade for the result and map that back to $\bigwedge \mathcal{V}^{\top}$ using the inverse of $\bigwedge g^{-1}$. So we will define $\widetilde{*}: \bigwedge \mathcal{V}^{\top} \to \bigwedge \mathcal{V}^{\top}$ as $(\bigwedge g^{-1})^{-1} \circ * \circ \bigwedge g^{-1}$, which works out to be $H \circ \bigwedge g^{-1}$ when the unscaled $\bigwedge g^{-1} \circ H$ is substituted for *. Thus, applying $\widetilde{*}$ to an exterior product of vectors from the dual basis \mathcal{B}_H^{\top} gives

 $\widetilde{*}\left(x_{i_1}^{\top} \wedge \cdots \wedge x_{i_n}^{\top}\right) = H\left(x_{i_1}^{\perp} \wedge \cdots \wedge x_{i_n}^{\perp}\right).$

Employing the usual standard scalar product with \mathcal{B}_H as its chosen basis, we then have, for example, $\widetilde{*}(x_1^\top \wedge \cdots \wedge x_n^\top) = x_{n+1}^\top \wedge \cdots \wedge x_d^\top$.

Exercise 10.18 Using \widetilde{g} to define orthogonality for \mathcal{V}^{\top} , $\widetilde{*}\beta$ is an orthogonal blade of the blade β in $\bigwedge \mathcal{V}^{\top}$.

As an alternative to the related Hodge star $\widetilde{*}$ that we have just defined, we could define a Hodge star on $\bigwedge \mathcal{V}^{\top}$ in the manner of the previous section, using the related scalar product \widetilde{g} on \mathcal{V}^{\top} extended to each $\bigwedge^p \mathcal{V}^{\top}$.

Exercise 10.19 Compare $\widetilde{*}$ with the Hodge star defined on $\bigwedge \mathcal{V}^{\top}$ in the manner of the previous section, using the related scalar product \widetilde{g} on \mathcal{V}^{\top} .

10.12 Problems

- 1. For finite-dimensional spaces identified with their double duals, the included maps belonging to a pairing are the duals of each other.
- 2. A pairing of a vector space with itself is reflexive if and only if it is either symmetric or alternating.
- 3. Give an example of vectors u, v, w such that $u \perp v$ and $v \perp w$ but it is not the case that $u \perp w$.
- 4. The pairing g of two d-dimensional vector spaces with respective bases $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$ is nondegenerate if and only if

$$\det \begin{bmatrix} g(x_1, y_1) & \cdots & g(x_1, y_d) \\ \vdots & \cdots & \vdots \\ g(x_d, y_1) & \cdots & g(x_d, y_d) \end{bmatrix} \neq 0.$$

- 5. A pairing g of two finite-dimensional vector spaces \mathcal{V} and \mathcal{W} with respective bases $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_e\}$ is an element of the functional product space $\mathcal{V}^{\top}\mathcal{W}^{\top}$ and may be expressed as $g = \sum_{i,j} g(x_i, y_j) \ x_i^{\top} \ y_j^{\top}$.
- 6. When \mathcal{F} is the field with just two elements, so that 1+1=0, any scalar product on \mathcal{F}^2 will make some nonzero vector orthogonal to itself.
- 7. For a scalar product space over a field where $1 + 1 \neq 0$, the scalar product g satisfies g(w v, w v) = g(v, v) + g(w, w) if and only if $v \perp w$.
- 8. What, if anything, is the Hodge star on $\bigwedge \mathcal{V}$ when dim $\mathcal{V} = 1$?
- 9. With the same setup as Exercise 10.17,

$$r = \sum_{|I|=p} (-1)^{\rho} g(*r, x_{\overline{I}}) \cdot x_I = \sum_{|J|=d-p} (-1)^{\overline{J}J} g(*r, x_J) \cdot x_{\overline{J}},$$

while

$$*(*r) = G^{-1} \sum_{|J|=d-p} (-1)^{J\overline{J}} g(*r, x_J) \cdot x_{\overline{J}}.$$

where $J = \{j_1, \dots, j_{d-p}\}, \overline{J} = \{j_{d-p+1}, \dots, j_d\}, \text{ and } J \cup \overline{J} = \{1, \dots, d\}.$ Hence

$$*(*r) = G^{-1}(-1)^{p(d-1)} \cdot r = G^{-1}(-1)^{p(d-p)} \cdot r.$$

- 10. $*^{-1}(*s \wedge *t) = H^{-1}(H(s) \wedge H(t)) = s \vee t \text{ for } s, t \in \bigwedge \mathcal{V}$. On the other hand, $\widetilde{*}(\widetilde{*}^{-1}\sigma \wedge \widetilde{*}^{-1}\tau) = H(H^{-1}(\sigma) \wedge H^{-1}(\tau)) = \sigma \widetilde{\vee} \tau \text{ for } \sigma, \tau \in \bigwedge \mathcal{V}^{\top}$. How does $\widetilde{\vee}$ compare to the regressive product defined on $\bigwedge \mathcal{V}^{\top}$ using the dual of the basis used for $\bigwedge \mathcal{V}$? How do $*(*s \wedge *t)$ and $\widetilde{*}(\widetilde{*}\sigma \wedge \widetilde{*}\tau)$ relate to the same respective regressive products?
- 11. In \mathbb{R}^3 with the standard basis used for defining H and as the chosen basis for the usual standard scalar product, the familiar cross product $u \times v$ of two vectors u and v is the same as $*(u \wedge v)$.
- 12. Using multi-index notation as in Section 10.9 above,

$$x_I \wedge *x_I = g(x_I, x_I) \cdot x_1^{\perp} \wedge \dots \wedge x_d^{\perp}$$

= $g(x_I, x_I) \cdot (G^{-1} \cdot x_1 \wedge \dots \wedge x_d)$

where $\{x_1, \ldots, x_d\}$ is the basis used in defining *, and $G = \det[g(x_i, x_j)]$.

13. Using $\{x_1, \ldots, x_d\}$ as the basis in defining *, and using multi-index notation as in Section 10.9 above, then

$$*x_J = G^{-1} \sum_{|I|=|J|} (-1)^{\rho} g(x_I, x_J) \cdot x_{\overline{I}},$$

but

$$\widetilde{*} x_J^{\top} = \sum_{|I|=|J|} (-1)^{\rho} \widetilde{g}(x_I, x_J) \cdot x_{\overline{I}}^{\top},$$

where $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_p\}$, $\overline{I} = \{i_{p+1}, \dots, i_d\}$ is such that $I \cup \overline{I} = \{1, \dots, d\}$, $\rho = \begin{pmatrix} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{pmatrix}$ and $G = \det[g(x_i, x_j)]$.

14. Let $\{x_1, \ldots, x_d\}$ be the basis used in defining *, let \mathcal{F} be the real numbers, and let $d = \dim \mathcal{V} > 1$. The symmetric bilinear functional g on \mathcal{V} such that

$$g(x_i, x_j) = \begin{cases} 1, \text{ when } i \neq j \\ 0, \text{ otherwise} \end{cases}$$

is a scalar product that makes $x_j \wedge *x_j = 0$ for each j. (The nondegeneracy of g follows readily upon observing that multiplying the matrix $[g(x_i, x_j)]$ on the right by the matrix $[h_{i,j}]$, where

$$h_{i,j} = \begin{cases} -1, \text{ when } i \neq j \\ d - 2, \text{ otherwise} \end{cases},$$

gives -(d-1) times the identity matrix.)

- 15. If two bases have the same Gram determinant with respect to a given scalar product, then they have the same Gram determinant with respect to any scalar product.
- 16. All chosen bases that produce the same positive standard scalar product also produce the same Hodge star up to sign.